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RATIONAL POINTS IN FUNCTION FIELDS

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Adelina Mânzăţeanu:

Rational points in function fields

ABSTRACT

A function field version of the circle method is applied to a cubic hypersurface X defined over a finite field $\mathbb{F}_q$. Using the correspondence between $\mathbb{F}_q$-rational curves and $\mathbb{F}_q(t)$-points, we deduce the dimension and irreducibility of the moduli space of rational curves on X passing through two fixed points. Furthermore, we study Manin's conjecture over function fields and obtain an example where the conjecture holds after removing a thin set of points. This leads to an application which can be seen as the prime number theorem for 0-cycles on $\mathbb{P}^2$.

To my parents.

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DECLARATION

I declare that the work in this dissertation was carried out in accordance with the requirements of the University's 'Regulations and Code of Practice for Research Degree Programmes' and that it has not been submitted for any other academic award. Except where indicated by specific reference in the text, the work is my own work. Work done in collaboration with, or with the assistance of, others is indicated as such. Any views expressed in this dissertation are those of the author.

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CONTENTS

1

INTRODUCTION

1.1 DIOPHANTINE EQUATIONS

The study of integer solutions of polynomial equations with integer coefficients dates back to Diophantus. Over time, this problem has been generalised to global fields K, that is, number fields (finite extensions of $\mathbb{Q}$) and global function fields (finite extensions of $\mathbb{F}_q(t)$, where $\mathbb{F}_q$ is a finite field). The solutions to such equations can be seen as rational points on some variety. In particular, let X be a variety in $\mathbb{A}^n_K$ defined by a system of polynomial equations

$$f_i(x_1, \ldots, x_n) = 0, \qquad 1 \leqslant i \leqslant r.$$

We are interested in the set

$$X(K) = \{\mathbf{a} \in K^n \mid f_1(\mathbf{a}) = \ldots = f_r(\mathbf{a}) = 0\}$$

of *K-rational points on X*. There are several questions that one would like to know the answer to, such as:

1. Is $X(K) \neq \varnothing$?

2. If $X(K) \neq \varnothing$, then is $X(K)$ finite or infinite?

3. If $X(K)$ is finite, then can its elements be listed?

4. If $X(K)$ is infinite, then can its size be measured? More precisely, by defining a height function on $H : X(K) \to \mathbb{R}_{\geqslant 0}$, can one predict the growth rate of

   $$N_X(B) := \#\{x \in X(K) : H(x) \leqslant B\}$$

   as $B \to \infty$?

5. Given an embedding of X in $\mathbb{A}^n$, then what can one say about degree d rational curves on X?

6. Given an embedding of X in $\mathbb{A}^n$, what can one say about higher genus curves on X?

The goal of this thesis is to study Questions 4 and 5 for certain varieties in the case when K is a global function field, while emphasising the relation between counting using analytic methods and the geometry of the varieties considered.

1.2 CONTENTS OF THE THESIS

In this section, we give a summary of the chapters of this work and briefly present some of the main results.

CHAPTER 2. PRELIMINARIES. In this chapter, we present the background material needed in this thesis and some of the previously known results. In Sections 2.1, 2.2 and 2.3, we recall some definitions and basic theory about global function fields. The analogy between integers and polynomials in one variable with coefficients in $\mathbb{F}_q$ is also explained. Section 2.4 is concerned with Manin's conjecture, which predicts the asymptotic behaviour of the number of rational points of bounded height for a certain class of varieties; hence, gives an expected answer to Question 4. More precisely, if X is a projective variety over a global field K, H a *height function* that takes values in $\mathbb{R}_{>0}$ and measures the "size" of a solution to the defining equations of X, then by the conjecture, we expect that

$$\# \{x \in X(K) : H(x) \leqslant B\} \sim cB^{\alpha} (\log B)^{\beta},$$

as $B \to \infty$. Furthermore, we describe Peyre's refinements to this conjecture and his geometric interpretation of the leading constant. Then, in Section 2.6, we introduce the Hardy–Littlewood circle method which is one of the main approaches that can be used to tackle Question 4 in the case when the dimension of the variety is large with respect to its degree. Throughout this chapter, we also give a comparison to the situation over number fields and discuss some of the advantages of working in the function field setting.

CHAPTER 3. RATIONAL CURVES ON CUBIC HYPERSURFACES. Let $X \subset \mathbb{P}^{n-1}$ denote a smooth cubic hypersurface defined over a finite field $\mathbb{F}_q$ of characteristic greater than 3, and a, b be two fixed $\mathbb{F}_q$-points on X, not both on the Hessian. Using a function field analogue of the Hardy–Littlewood circle method, we count polynomials $f_1, \ldots, f_n \in \mathbb{F}_q[t]$ of degree at most d whose constant and leading coefficients are given by $\mathbf{a}, \mathbf{b} \in \mathbb{F}_q^n$, corresponding to the points a, b, such that $F(f_1, \ldots, , f_n) = 0$. This corresponds to the number $N_{a,b}(d)$ of $\mathbb{F}_q(t)$-points of bounded height on X which satisfy certain additional conditions. We obtain the following asymptotic formula.

Theorem 1.2.1. *Fix $k = \mathbb{F}_q$ with $\mathrm{char}(k) > 3$. Fix a smooth cubic hypersurface $X \subset \mathbb{P}_k^{n-1}$, where $n \geqslant 10$. Let $a, b \in X(k)$, not both on the Hessian. Then, we have*

$$N_{a,b}(d) = q^{(d-1)n-(3d-1)} + O\left(q^{\frac{5(d+2)n}{6} - \frac{5(d+2)}{3}} + q^{\frac{(5d+8)n}{6} - \frac{3d}{2} - \frac{8}{3}} + q^{\frac{3(d+5)n}{4} - \frac{3d+7}{4}}\right),$$

where the implied constant in the estimate depends only on d and X.

Using the correspondence between $\mathbb{F}_q$-rational curves of bounded degree and the $\mathbb{F}_q(t)$-points of bounded height on a variety, we use Theorem 1.2.1 to study the geometry of the moduli space of rational curves of a fixed degree d on X passing through two fixed points. This allows us to deduce its dimension and number of irreducible components, provided d is large enough. What is more, the result over $\mathbb{F}_q$ can be extended to one over $\mathbb{C}$ via a "spreading out" argument similar to the one in [16] which is described in Section 3.1.

Corollary 1.2.2. *Fix a smooth cubic hypersurface $X \subset \mathbb{P}^{n-1}$ defined over $\mathbb{C}$, where $n \geqslant 10$. Pick any points two points in $X(\mathbb{C})$, not both lying on the Hessian. Then, for each $d \geqslant \frac{19n-11}{n-9}$, the space of morphisms of degree d from $\mathbb{P}_{\mathbb{C}}^1 \to X$ passing through two fixed points, denoted $\mathcal{M}_{0,2}(\mathbb{P}_{\mathbb{C}}^1, X, d)$, is irreducible and of expected dimension $(n-3)d - n - 3$.*

CHAPTER 4. MANIN'S CONJECTURE FOR $\mathrm{Hilb}^2 \mathbb{P}^2$. Over number fields, there are several counterexamples to Manin's conjecture which led to a refinement where one is allowed to eliminate a thin set in the sense of Serre [75, §3.1]. If V is an irreducible algebraic variety defined over a field K, a *thin* set T is a set that is contained in a finite union of sets of the type (C_1) or (C_2), where a subset $A \subseteq V(K)$ is of type (C_1) if A is not Zariski-dense in V, and of type (C_2) if there exists an irreducible variety W of V of dimension equal to $\dim V$ and a generically surjective morphism $\pi : W \to V$ of degree $\geqslant 2$ with $A \subset \pi(W(K))$. Type (C_1) sets include the case when the exceptional set is contained in a proper closed subset and the conjecture is known for several examples, such as flag varieties [27], toric varieties [5], complete intersections in many variables [8, 65]. Browning–Heath-Brown [14] have studied an example in which the thin set is of type (C_2). In particular, they consider the hypersurface $X \subset \mathbb{P}^3 \times \mathbb{P}^3$ defined over $\mathbb{Q}$ by $x_1 y_1^2 + \ldots + x_4 y_4^2 = 0$ and eliminate the thin set $\{(x, y) \in X(\mathbb{Q}) : x_1 \ldots x_4 = \square\}$. Inspired by work of Le Rudulier [58] over $\mathbb{Q}$, we study the Hilbert scheme of two points on $\mathbb{P}^2$ defined over a global field K of characteristic greater than 2. This variety is precisely the desingularisation of the symmetric product $\operatorname{Sym}^2 \mathbb{P}^2 \cong \mathbb{P}^2 \times \mathbb{P}^2 / \mathfrak{S}_2$, where $\mathfrak{S}_2$ is the symmetric group of 2 elements and acts on $\mathbb{P}^2 \times \mathbb{P}^2$ by permuting the factors. We give an asymptotic formula for the number of K-points of bounded height on $\operatorname{Hilb}^2 \mathbb{P}^2$ and show that by eliminating an exceptional thin set, the refined version of Manin's conjecture holds. This is the first result over global function fields supporting the thin set version of Manin's conjecture. In particular, we prove the following result.

Theorem 1.2.3. *There exists a non-empty thin set $Z_0 \subset \operatorname{Hilb}^2 \mathbb{P}^2(K)$ such that*

$$\#\left\{z \in \operatorname{Hilb}^2 \mathbb{P}^2(K) \backslash Z_0 : H_{\omega^{-1}_{\operatorname{Hilb}^2 \mathbb{P}^2}}(z) = q^M\right\} = c q^M M + O\left(\sqrt{M} q^M\right),$$

as $M \to \infty$, where the leading constant agrees with the prediction of Peyre in [65].

CHAPTER 5. A VERSION OF THE PRIME NUMBER THEOREM FOR 0-CYCLES. In this chapter, we recall the analogy between positive integers and effective 0-cycles on a variety V over $\mathbb{F}_q$, i.e. formal sums $C = \sum_{i=1}^k n_i P_i$, where $n_i \in \mathbb{N}$ and P_i are distinct closed points on V. In particular, primes correspond to the points P_i. In Section 5.1, we extend this analogy to effective 0-cycles on a variety V over a global field K of positive characteristic and explain how 0-cycles relate to K-points on symmetric products. This allows us to establish a version of the prime number theorem in the case when $V = \mathbb{P}^2$.

Theorem 1.2.4. *Let $m \geqslant 2$ and K be a global function field of characteristic $> m$. Suppose that Manin's conjecture holds for the irreducible points in $\operatorname{Hilb}^{m_0} \mathbb{P}^2(K)$ for all $m_0 \leqslant m$. Then, there exists a constant $c_m > 0$ such that the proportion of effective 0-cycles on $\mathbb{P}^2$ over K of degree m corresponding to K-points on $\operatorname{Sym}^m(\mathbb{P}^2)$ of height q^M that are prime is*

$$\sim \frac{c_m}{M^{m-2}},$$

$M \to \infty$.

In general, c_m is quite complicated, but an explicit expression is given in Corollary 5.1.2. When $m = 2$, however, $c_2 = \frac{2}{3}$ and the result is actually unconditional.

2

PRELIMINARIES

In this chapter, we will introduce some of the prerequisites for the later parts of the thesis. We will start by presenting the well-known analogy between the rationals and the rational function field $\mathbb{F}_q(t)$. Then, we will introduce Manin's conjecture, its function field version and its refinements, as these are the root of motivation for the work in Chapter 4. In the end, we will describe the Hardy–Littlewood circle method which is used in Chapter 3 and is one of the main approaches that can be used to prove the aforementioned conjecture.

2.1 BASIC NOTIONS ON GLOBAL FUNCTION FIELDS

Many problems in classical number theory are concerned with the study of the properties of integer numbers. The similarity between the structure of $\mathbb{Z}$ and that of the ring of polynomials in one variable with coefficients in a finite field $\mathbb{F}_q$ led to posing analogue questions over $\mathbb{F}_q[t]$. We give an exposition of some of these analogies, but we also refer the reader to [30]. The standard convention $e(x) = e^{2\pi i x}$ is used.

The corresponding fields of fractions of $\mathbb{Z}$ and $\mathcal{O} := \mathbb{F}_q[t]$ are $\mathbb{Q}$ and the rational function field $K := \mathbb{F}_q(t)$, respectively. The non-archimedean places of K are monic irreducible polynomials $\varpi = t^n + a_{n-1}t^{n-1} + \ldots + a_0$, with all $a_i \in \mathbb{F}_q$, which correspond to primes p in $\mathbb{Q}$. To these, we have the associated non-archimedean absolute values $|\cdot|_\varpi = \left(\frac{1}{q^{\deg(\varpi)}}\right)^{\operatorname{ord}_\varpi(\cdot)}$, and $|\cdot|_p = p^{-\operatorname{val}_p(\cdot)}$, respectively, where $\deg(\varpi)$ is the degree of the polynomial $\varpi \in \mathbb{F}_q[t]$, $\operatorname{ord}_\varpi$ is the discrete valuation of $\mathbb{F}_q(t)$ associated to the place ϖ and val_p is the discrete valuation of $\mathbb{Q}$ associated to the place p. The archimedean place of $\mathbb{Q}$ corresponds to the real absolute value, whereas in the case of K, we let $s := t^{-1}$ be the "place at infinity" and this corresponds to the non-archimedean absolute value

$|\cdot|_\infty = q^{\deg(\cdot)}$ on $\mathcal{O}$. This can be extended to K by $\left|\frac{a}{b}\right|_\infty = q^{\deg a - \deg b}$, where $a, b \in \mathcal{O}$. We obtain the following completions with respect to the absolute values

$$\mathbb{Q}_p := \operatorname{Frac} \mathbb{Z}_p = \left\{ \sum_{i=i_0}^{\infty} a_i p^i | a_i \in \mathbb{F}_p, i_0 \in \mathbb{Z} \right\} \quad \text{and} \quad \mathbb{R},$$

and

$$K_\varpi := \mathbb{F}_q((t)) = \operatorname{Frac} \mathbb{F}_q[[t]] = \left\{ \sum_{i=i_0}^{\infty} a_i t^i | a_i \in \mathbb{F}_q, i_0 \in \mathbb{Z} \right\},$$

$$K_\infty := \mathbb{F}_q\left(\left(\frac{1}{t}\right)\right) = \left\{ \sum_{i \leqslant N} a_i t^i | a_i \in \mathbb{F}_q, N \in \mathbb{Z} \right\},$$

respectively. In the case of non-archimedean fields, the local rings of integers are the subrings containing elements of norm $\leqslant 1$. Thus, in this case, the corresponding local rings are

$$\mathbb{Z}_p := \varprojlim_{n \in \mathbb{N}} \frac{\mathbb{Z}}{(p^n)} = \left\{ \sum_{i=0}^{\infty} a_i p^i | a_i \in \mathbb{F}_p \right\},$$

and

$$\mathcal{O}_\varpi := \mathbb{F}_q[[t]] = \varprojlim_{n \in \mathbb{N}} \frac{\mathbb{F}_q[t]}{(t^n)} = \left\{ \sum_{i=0}^{\infty} a_i t^i | a_i \in \mathbb{F}_q \right\},$$

$$\mathcal{O}_\infty := \left\{ \sum_{i \leqslant 0} a_i t^i | a_i \in \mathbb{F}_q \right\},$$

respectively.

Since some of our methods are based on Fourier analysis on function fields, it is necessary to introduce the torus

$$\mathbb{T} := \left\{ \sum_{i \leqslant -1} a_i t^i | a_i \in \mathbb{F}_q \right\},$$

which corresponds to the interval $[0, 1)$.

Definition 2.1.1. Let e_q be a non-trivial additive character on $\mathbb{F}_q$ defined by

$$\begin{aligned} e_q : \mathbb{F}_q &\to \mathbb{C}^* \\ a &\mapsto e\left(\frac{\operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(a)}{p}\right), \end{aligned}$$

where $\operatorname{Tr}_{\mathbb{F}_{p^m}/\mathbb{F}_p}(a) = \sum_{\sigma \in \operatorname{Gal}(\mathbb{F}_{p^m}/\mathbb{F}_p)} \sigma(a) = a + a^p + a^{p^2} + \ldots + a^{p^{m-1}}$. Then, we define a non-trivial character on K_∞ given by

$$\begin{aligned} \psi : K_\infty &\to \mathbb{C}^* \\ \sum_{i \leqslant N} a_i t^i &\mapsto e_q(a_{-1}), \end{aligned}$$

where a_{-1} is the coefficient of t^{-1}.

We remark that the character ψ defined above is a non-trivial additive character since

$$\psi\left(\sum_{i\leqslant N} a_i t^i + \sum_{i\leqslant M} b_i t^i\right) = e_q\left(a_{-1}+b_{-1}\right) = e_q\left(a_{-1}\right)e_q(b_{-1}) = \psi\left(\sum_{i\leqslant N} a_i t^i\right)\psi\left(\sum_{i\leqslant M} b_i t^i\right).$$

Moreover, for any $\alpha \in \mathbb{F}_q[t]$, the coefficient of t^{-1} is 0; hence, $\psi|_{\mathbb{F}_q[t]} = e_q(0) = 1$. Thus, we have the following corresponding additive characters

$$\begin{aligned} e : [0,1) &\to \mathbb{C}^* \\ x &\mapsto e^{2\pi i x} \end{aligned} \qquad \text{and} \qquad \begin{aligned} \psi : \mathbb{T} &\to \mathbb{C}^* \\ \sum_{i\leqslant -1} a_i t^i &\mapsto e_q(a_{-1}). \end{aligned}$$

Locally compact topological spaces have Haar measures, which implies that there is a (Haar) measure on K_∞, and so on $\mathbb{T}$. We normalise it such that $\int_{\mathbb{T}} \mathrm{d}\alpha = 1$. This way, the property that

$$\int_0^1 e(\alpha m)\,\mathrm{d}\alpha = \begin{cases} 0, & \text{if } m \in \mathbb{Z}^*, \\ 1, & \text{if } m = 0 \end{cases} \tag{2.1}$$

corresponds to

$$\int_{\mathbb{T}} \psi(\alpha\gamma)\,\mathrm{d}\alpha = \begin{cases} 0, & \text{if } \gamma \in \mathbb{F}_q[t]\backslash\{0\}, \\ 1, & \text{if } \gamma = 0. \end{cases}$$

Note that the normalisation we have chosen implies that

$$\int_{\{\alpha\in K_\infty : |\alpha| < \widehat{N}\}} \mathrm{d}\alpha = q^N,$$

for any positive integer N. This can be extended to $\mathbb{T}^n$ and K_∞^n for any $n \in \mathbb{Z}_{>0}$. Throughout this thesis, for any real number R, let $\widehat{R} = q^R$. The following orthogonality property in [52, Lemma 7] holds.

Lemma 2.1.2. *For any $N \in \mathbb{Z}_{\geqslant 0}$ and any $\gamma \in K_\infty$, we have*

$$\sum_{\substack{b\in\mathbb{F}_q[t] \\ |b|<\widehat{N}}} \psi(\gamma b) = \begin{cases} \widehat{N}, & \textit{if } |\gamma| < \widehat{N}^{-1}, \\ 0, & \textit{else}, \end{cases}$$

where $|\cdot|$ is the absolute value corresponding to the place at infinity.

The following lemma corresponds to [15, Lemma 2.2] and its proof can also be found in [52, Lemma 1(f)].

Lemma 2.1.3. *Let $Y \in \mathbb{Z}$ and $\gamma \in K_\infty$. Then*

$$\int_{|\alpha|<\widehat{Y}} \psi(\alpha\gamma)\,\mathrm{d}\alpha = \begin{cases} \widehat{Y}, & \text{if } |\gamma| < \widehat{Y}^{-1}, \\ 0, & \text{otherwise.} \end{cases}$$

Taking $Y = 0$, it follows that

$$\int_{\alpha\in\mathbb{T}} \psi(\alpha\gamma)\,\mathrm{d}\alpha = \begin{cases} 1, & \text{if } \gamma = 0, \\ 0, & \text{if } \gamma \in \mathcal{O}\backslash\{0\}. \end{cases}$$

2.2 ANALYTIC FUNCTIONS

This section is concerned with polynomial analogues of various analytic functions that are used in the later chapters of this work.

Definition 2.2.1. We define the polynomial Möbius function $\mu : \mathbb{F}_q[t] \to \{0, \pm 1\}$ to be given by

$$\mu(f) = \begin{cases} 0, & \text{if } f \text{ is not square-free}, \\ (-1)^l, & \text{if } f = cp_1 \dots p_l, \end{cases}$$

where $c \in \mathbb{F}_q$ and $p_1, \dots, p_l \in \mathbb{F}_q[t]$ are l monic distinct irreducible polynomials.

Therefore, we have the following two properties of which we will make use repeatedly in a step called *Möbius inversion* that appears in the counting arguments in the chapters to follow. In particular, in Section 2.5.1 we give a detailed account of the process.

Lemma 2.2.2. *Let $g \in \mathbb{F}_q[t]$. We have*

$$\sum_{\substack{k\in\mathbb{F}_q[t] \\ \text{monic} \\ k|g}} \mu(k) = \begin{cases} 1, & \text{if } g = 1, \\ 0, & \text{if } g \neq 1. \end{cases}$$

Proof. The result is clear if $g = 1$, since this implies $k = 1$ and $\mu(1) = 1$. Assume $g \neq 1$ and write $g = cp_1^{\alpha_1} \dots p_l^{\alpha_l}$, where $c \in \mathbb{F}_q$, $p_i \in \mathbb{F}_q[t]$ are monic and irreducible polynomials and $\alpha_i \in \mathbb{N}$ for all $1 \leqslant i \leqslant l$. Then, by Definition 2.2.1, we obtain

$$\sum_{\substack{k\in\mathbb{F}_q[t] \\ \text{monic} \\ k|g}} \mu(k) = \mu(1) + \sum_{i=1}^{l} \mu(p_i) + \sum_{\substack{1\leqslant i,j\leqslant l \\ i\neq j}} \mu(p_i p_j) + \dots + \mu(p_1 \dots p_l) = (1 + (-1))^l = 0,$$

where the penultimate equality follows from the binomial theorem. □

Lemma 2.2.3. *We have*

$$\sum_{\substack{k \in \mathbb{F}_q[t] \\ \text{monic} \\ |k| = q^j}} \mu(k) = \begin{cases} 1, & \text{if } j = 0, \\ -q, & \text{if } j = 1, \\ 0, & \text{if } j > 1. \end{cases} \tag{2.2}$$

Proof. In the case when $j = 0$, the sum on the left hand-side of (2.2) becomes a sum over monic $k \in \mathbb{F}_q$. Thus, $k = 1$ and $\mu(1) = 1$. When $j = 1$, we obtain

$$\sum_{\substack{k \in \mathbb{F}_q[t] \\ \text{monic} \\ |k| = q}} \mu(k) = \sum_{k' \in \mathbb{F}_q} \mu(t + k') = \sum_{k' \in \mathbb{F}_q} -1 = -q.$$

We are left with the case when $j > 1$. As defined in [70, Chapter 2], the zeta function of $\mathbb{F}_q[t]$ is

$$\zeta_{\mathbb{F}_q[t]}(s) = \sum_{\substack{k \in \mathbb{F}_q[t] \\ \text{monic}}} \frac{1}{|k|^s}.$$

Since there are q^n monic polynomials $k \in \mathbb{F}_q[t]$ of degree n, we obtain

$$\zeta_{\mathbb{F}_q[t]}(s) = \sum_{n \geqslant 0} \frac{q^n}{q^{ns}} = \frac{1}{1 - q^{1-s}} = \prod_{\substack{p \in \mathbb{F}_q[t] \\ \text{monic} \\ \text{irreducible}}} \left(1 - \frac{1}{|p|^s}\right)^{-1}, \tag{2.3}$$

for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, where the last equality follows from the unique factorisation of monic polynomials into monic irreducible polynomials. Since the Möbius function is multiplicative, we have the following Euler product

$$\sum_{\substack{k \in \mathbb{F}_q[t] \\ \text{monic}}} \frac{\mu(k)}{|k|^s} = \prod_{\substack{p \in \mathbb{F}_q[t] \\ \text{monic} \\ \text{irreducible}}} \left(1 + \frac{\mu(p)}{|p|^s} + \frac{\mu(p^2)}{|p|^{2s}} + \dots\right) = \prod_{\substack{p \in \mathbb{F}_q[t] \\ \text{monic} \\ \text{irreducible}}} \left(1 - \frac{1}{|p|^s}\right), \tag{2.4}$$

which is precisely $\zeta_{\mathbb{F}_q[t]}(s)^{-1}$ by (2.3). Thus, we have

$$\sum_{n \geqslant 0} \frac{1}{q^{ns}} \sum_{\substack{k \in \mathbb{F}_q[t] \\ \text{monic} \\ |k| = q^n}} \mu(k) = 1 - q^{1-s}.$$

Setting $q^{-s} = u$ and equating coefficients leads to

$$\sum_{n \geqslant 2} \sum_{\substack{k \in \mathbb{F}_q[t] \\ \text{monic} \\ |k| = q^n}} \mu(k) u^n = 0,$$

as claimed. □

The next three results are standard, but are proved here since in Chapter 3 we require versions in which the implied constant is independent of q.

Lemma 2.2.4. *Let $\tau(f)$ be the number of monic divisors of a polynomial $f \in \mathbb{F}_q[t]$. Then, for any $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$, depending only on ε, such that $\tau(f) \leqslant C(\varepsilon)|f|^{\varepsilon}$.*

Proof. First note that

$$\frac{\tau(f)}{|f|^{\varepsilon}} = \prod_{\varpi^{\alpha}||f} \frac{\alpha+1}{|\varpi^{\alpha}|^{\varepsilon}} = \prod_{\substack{\varpi^{\alpha}||f \\ |\varpi|<2^{1/\varepsilon}}} \frac{\alpha+1}{|\varpi|^{\alpha\varepsilon}} \prod_{\substack{\varpi^{\alpha}||f \\ |\varpi|\geqslant 2^{1/\varepsilon}}} \frac{\alpha+1}{|\varpi|^{\alpha\varepsilon}},$$

where ϖ denotes a prime in $\mathcal{O}$. The second factor is less than or equal to 1. In the first factor, we have $|\varpi| < 2^{1/\varepsilon}$, which is equivalent to $d := \deg(\varpi) < \frac{1}{\varepsilon}\frac{\log 2}{\log q} =: D$. Then,

$$\prod_{\substack{\varpi^{\alpha}||f \\ |\varpi|<2^{1/\varepsilon}}} \frac{\alpha+1}{|\varpi|^{\alpha\varepsilon}} = \prod_{d<D} \prod_{\substack{\varpi^{\alpha}||f \\ |\varpi|=q^d}} \frac{\alpha+1}{q^{d\alpha\varepsilon}} \leqslant \prod_{d<D} \prod_{\substack{\varpi^{\alpha}||f \\ |\varpi|=q^d}} \left(1 + \frac{\alpha}{q^{d\alpha\varepsilon}}\right).$$

Now, if $g(\alpha) = \frac{\alpha}{y^{\alpha}}$, then $g'(\alpha) = y^{-\alpha}(1 - \alpha \log y)$. Thus, g is maximised at $\alpha = \frac{1}{\log y}$ when $g\left(\frac{1}{\log y}\right) = \frac{1}{e \log y}$. Let a_d denote the number of primes of degree d in $\mathbb{F}_q[t]$. Hence,

$$\prod_{\substack{\varpi^{\alpha}||f \\ |\varpi|<2^{1/\varepsilon}}} \frac{\alpha+1}{|\varpi|^{\alpha\varepsilon}} \leqslant \prod_{d<D} \prod_{|\varpi|=q^d} \left(1 + \frac{1}{e \log q^{d\varepsilon}}\right) \leqslant \prod_{d<D} \left(1 + \frac{1}{e \log q^{d\varepsilon}}\right)^{\frac{2q^d}{d}},$$

since by [70, Chapter 2], we have

$$\left| a_d - \frac{q^d}{d} \right| \leqslant \frac{q^{\frac{d}{2}}}{d} + q^{\frac{d}{3}}. \tag{2.5}$$

Then, using $1 + x \leqslant e^x$, we obtain

$$\prod_{\substack{\varpi^{\alpha}||f \\ |\varpi|<2^{1/\varepsilon}}} \frac{\alpha+1}{|\varpi|^{\alpha\varepsilon}} \leqslant \prod_{d<D} \left(\exp\left(\frac{1}{e \log q^{d\varepsilon}}\right)\right)^{\frac{2q^d}{d}} = \exp\left(2 \sum_{d<D} \frac{q^d}{d^2} \cdot \frac{1}{e\varepsilon \log q}\right).$$

Now, q^d/d^2 is increasing with d for $q \geqslant 4$ and thus, in this case we have $\sum_{d<D} q^d/d^2 < q^D/D$. In fact, we have $\sum_{d<D} q^d/d^2 < 2q^D/D$, for any $q \geqslant 2$. Thus,

$$\prod_{\substack{\varpi^{\alpha}||f \\ |\varpi|<2^{1/\varepsilon}}} \frac{\alpha+1}{|\varpi|^{\alpha\varepsilon}} \leqslant \exp\left(\frac{4q^D}{D} \cdot \frac{1}{e\varepsilon \log q}\right) = \exp\left(\frac{2^{2+1/\varepsilon}}{e \log 2}\right),$$

which concludes the proof. □

Lemma 2.2.5. *Let $\omega(f)$ denote the number of prime divisors of a polynomial $f \in \mathbb{F}_q[t]$. Then for any $\varepsilon > 0$ and any integer $k \geqslant 2$, we have $k^{\omega(f)} \ll_{\varepsilon,k} |f|^{\varepsilon}$.*

Proof. Let $\tau_k(f)$ denote the number of factorisations of a polynomial $f \in \mathbb{F}_q[t]$ into k factors. Write $f = \varpi_1^{a_1} \dots \varpi_m^{a_m}$, where ϖ_i are distinct primes in $\mathbb{F}_q[t]$. Then,

$$\tau_k(f) = \prod_{j=1}^{m} \binom{a_j + k - 1}{a_j} = \prod_{j=1}^{m} \frac{(a_j + k - 1) \dots (a_j + 1)}{(k-1)!} \geqslant \prod_{j=1}^{m} \frac{k!}{(k-1)!} = k^m.$$

Thus, $\tau_k(f) \geqslant k^{\omega(f)}$. We will prove $\tau_k(f) \ll_\varepsilon |f|^\varepsilon$, by induction. For $k = 2$, the result follows from Lemma 2.2.4. For $k > 2$, use the fact that $\tau_k(f) = \sum_{d|f} \tau_{k-1}(f/d_k)$. $\square$

2.3 PRODUCT FORMULA

Another important property that both $\mathbb{Q}$ and $\mathbb{F}_q(t)$ share is that they satisfy a product formula with respect to their sets of absolute values. We will describe this in higher generality. Throughout this section, let K be a global field of degree $e \geqslant 1$, i.e. $[K : F] = e$, where $F = \mathbb{Q}$, if K is a number field, and $F = \mathbb{F}_q(t)$, if K is a function field. Recall that two absolute values $|\ |$ and $|\ |'$ on K are *equivalent* if there exists $a \in \mathbb{R}_{>0}$ such that $|x|' = |x|^a$, for all $x \in K$, and an absolute value $|\ |$ on K is said to be *trivial* if $|x| = 1$ for all $x \in K - \{0\}$. Then, a *place* of K is an equivalence class of non-trivial absolute values. Let Ω_K denote the set of places of K. If $v \in \Omega_K$ is a place of K, then K_v is the completion of K at v, κ_v is the residue field of K_v and let $|\cdot|_v = (\#\kappa_v)^{-\operatorname{ord}_v(\cdot)}$ be the absolute value at v, where ord_v is the discrete valuation of F associated to v. We now define the *normalised absolute value* $\|\ \|_v : K_v \to \mathbb{R}_{\geqslant 0}$ on K_v to be equal to $|\ |_v$, if v is non-archimedean, the usual absolute value $|\ |_{\mathbb{R}}$ on $\mathbb{R}$, if v is a real place, and $|\ |_{\mathbb{C}}^2$, if v is complex. With this normalisation, the following result holds. This will be used later in this chapter and a proof can be found in [2, Lemma 6].

Lemma 2.3.1 (Product formula)**.** *Let K be a global field. For every $x \in K^\times$, we have*

$$\prod_{v \in \Omega_K} \|x\|_v = 1.$$

Extending the absolute values to K^n leads us to introducing *height* functions for the field K. Let $\mathbf{x} = (x_1, \dots, x_n) \in K^n$. We define

$$\|\mathbf{x}\|_v = \max\left(\|x_1\|_v, \dots, \|x_n\|_v\right).$$

Thus, if $\mathbf{x} \neq \mathbf{0}$, we have $|\mathbf{x}|_v = 1$ for all but finitely many v, and we define the *field height* of $\mathbf{x}$ by

$$H_K(\mathbf{x}) = \prod_{v \in \Omega_K} \|\mathbf{x}\|_v.$$

Now, by the product formula, we notice that

$$H_K(\lambda \mathbf{x}) = \prod_{v \in \Omega_K} \|\lambda \mathbf{x}\|_v = \prod_{v \in \Omega_K} \|\mathbf{x}\|_v = H_K(\mathbf{x}),$$

for any $\lambda \in K^\times$; hence, H_K is a function on $\mathbb{P}^{n-1}(K)$. A further discussion on height functions can be found in Section 2.5.

2.4 MANIN'S CONJECTURE

Let K be a global field and X a projective variety defined over K. We are interested to study the set of *rational points* $X(K)$. A point in this set is a solution to a system of diophantine equations and we can introduce a *height function* H, taking values in $\mathbb{R}_{>0}$, to measure its complexity. Then, for all $B \in \mathbb{R}$, we define

$$N_X(B) = \#\{x \in X(K) : H(x) \leqslant B\}.$$

Example 2.4.1. If $X = \mathbb{P}^n$ and $K = \mathbb{Q}$, then any rational point $x \in \mathbb{P}^n(\mathbb{Q})$ is an $n+1$-tuple of rational numbers, not all zero. Multiplying through by a common denominator and then diving by the greatest common divisor of the coordinates, we may assume that x is an $n+1$-tuple of coprime integers. Since multiplying through by -1 would still describe the same point, then it is natural to introduce the *exponential height* of x by setting

$$H_{\mathbb{Q}}(x) = \max(|x_0|, \ldots, |x_n|).$$

By Northcott's theorem, the set of $\mathbb{Q}$-rational points in $\mathbb{P}^n$ of bounded height $H_{\mathbb{Q}}$ is finite; hence, $N_{\mathbb{P}^n}(B)$ is bounded. By counting primitive lattice points in a ball, one obtains the classical asymptotic result

$$N_{\mathbb{P}^n}(B) \sim \frac{2^n}{\zeta(n+1)} B^{n+1},$$

as $B \to \infty$, where ζ is the Riemann zeta function.

We would like to extend the finiteness of $N_X(B)$ to more general projective varieties $X \subset \mathbb{P}^n$ defined over a number field K. First, we may extend the definition in Example 2.4.1 of the exponential height to that of a point $x \in \mathbb{P}^n(K)$ by setting

$$H_K(x) = \prod_{v \in \Omega_K} \max(|x_0|_v, \ldots, |x_n|_v),$$

where Ω_K is the set of non-trivial normalised valuations of K, as in Section 2.3. Then, for any finite extension L of K, we have that

$$H_L(x) = H_K(x)^{[L:K]}.$$

These are well-defined functions on the projective space, since number fields satisfy the product formula given by Lemma 2.3.1.

In order to define a height for a general projective variety X, we need to consider its embeddings in $\mathbb{P}^n$. As described in [18, Section 2.3], morphisms $X \to \mathbb{P}^n$ correspond to line bundles L on X together with $n+1$ sections that do not vanish identically. Moreover, it can be shown that the set of isomorphism classes of line bundles on X has an abelian group structure. This is precisely the *Picard group* $\mathrm{Pic}(X)$ of X. We give an alternative description of this group in the case when X is smooth. Define a *divisor* on X as a finite formal linear combination $D = \sum_i n_i P_i$, where $n_i \in \mathbb{Z}$ and P_i are prime divisors, i.e. irreducible closed subvarieties of X of codimension one defined over K. The divisors on X form a group denoted $\mathrm{Div}(X)$. Let f be a rational function on X. Then, to any prime divisor P we can associate a discrete valuation v_P and define the divisor of f to be $\mathrm{div}(f) = \sum_i v_{P_i}(f)P_i$. Such divisors are called *principal divisors* and they form a subgroup $\mathrm{Prin}(X)$ of $\mathrm{Div}(X)$. Then the Picard group of X is defined as the quotient $\mathrm{Div}(X)/\mathrm{Princ}(X)$.

Tensoring $\mathrm{Pic}(X)$ with $\mathbb{R}$, we obtain a real vector space $\mathrm{Pic}(X)_{\mathbb{R}}$. This contains the *effective cone* C_{eff} which is the cone generated by line bundles with non-zero sections over X. It can also be seen as the cone generated by the classes of all effective divisors, that is, divisors $D = \sum_i n_i P_i$ with all $n_i \geqslant 0$. Another important geometric object is the *ample cone* C_{ample} which is contained in C_{eff} and is generated by *very ample* line bundles, i.e. those line bundles whose corresponding morphism is an embedding of $\phi : X \hookrightarrow \mathbb{P}^n$. In the case when X is a smooth, one can define a special line bundle ω_X called the *canonical line bundle*. We will not define this notion here and we refer the reader to [39, Sections II.6 – II.8]. We call ω_X^{-1} the *anticanonical line bundle* of X.

For example, let X be a smooth degree d hypersurface in $\mathbb{P}^n$ given by $\{F = 0\}$. If $n \geqslant 3$, then $\mathrm{Pic}(X) \cong \mathbb{Z}\mathcal{O}_X(1)$ and $C_{\text{eff}} = C_{\text{ample}} = \mathbb{R}_{\geqslant 0}\mathcal{O}_X(1)$. Its canonical bundle is $\omega_X \cong (d - (n+1))\mathcal{O}_X(1)$ which is ample if $d > n+1$.

To an embedding $\phi : X \hookrightarrow \mathbb{P}^n$, we can associate a height function $H_\phi : X(\overline{K}) \to \mathbb{R}$ given by

$$H_\phi(x) = H \circ \phi(x). \tag{2.6}$$

Northcott showed that $N_X(B)$ is bounded if the chosen height H corresponds to an *ample* line line bundle, i.e. a line bundle with a very ample power.

The expected asymptotic behaviour of $N_X(B)$ as $B \to \infty$ has been predicted by Manin and his collaborators (see [3], [27]) by connecting the arithmetic of rational points and their heights with the geometry of the variety and its line bundles. This will briefly be described below.

Definition 2.4.2. Let X be a projective variety defined over a number field K, L an ample line bundle of X and $\phi : X \to \mathbb{P}^n$ the corresponding morphism. For $s \in \mathbb{C}$ we define

$$Z_X(s) = \sum_{x \in X(K)} \frac{1}{H_\phi(x)^s},$$

to be the *height zeta function*, where the height H_ϕ is as in (2.6).

In the case when Z_X has a meromorphic continuation with a unique largest pole at $s = a$ of order b, the Ikehara Tauberian theorem implies that

$$N_X(B) \sim cB^a (\log B)^b,$$

as $B \to \infty$. However, understanding the properties of the height zeta function is not a simple task in general.

Returning to geometry, it turns out that the behaviour of $N_X(B)$ is related to the class of the anticanonical line bundle ω_X^{-1} on X. Let us consider, for example, the case when X is a smooth curve in $\mathbb{P}^2$. If the genus of X is 0, then ω_X^{-1} is ample and either $X(K) = \varnothing$ or $X \cong \mathbb{P}^1$ in which case X is a rational curve and $N_X(B) \sim cB^2$. Otherwise, if the genus of X is 1, then X is an elliptic curve which has trivial canonical class. Then, by the Theorem of Mordell-Weil, we have that $N_X(B) \sim c(\log B)^{r/2}$, where r is the rank of X. Finally, if X is of genus $\geqslant 2$, then ω_X is ample and $X(K)$ is finite, by Falting's Theorem [23]. The higher dimensional analogues of these three cases are Fano varieties, whose anticanonical line bundle is ample, varieties with trivial canonical class, such as K3 surfaces, abelian varieties and Calabi-Yau varieties, and varieties of general type, which have ample canonical line bundle. Of particular interest is the former class of varieties, for which we have the following conjecture.

Conjecture 2.4.3 (Batyrev–Manin)**.** *If X is a Fano variety defined over a number field K, then there exists an open subset U of X such that*

$$\#\left\{x \in U(K) : H_{\omega_X^{-1}}(x) \leqslant B\right\} \sim c_{H_{\omega_X^{-1}}}(X) B (\log B)^{r_X - 1},$$

as $B \to \infty$, where $H_{\omega_X^{-1}}$ is the height associated to the anticanonical line bundle ω_X^{-1} and r_X is the rank of the Picard group of X.

Since there may be accumulating subvarieties of X that may dominate $N_X(B)$, considering the points in these subvarieties would not reflect the global distribution of rational points on X. However, an open set U is complementary to all these accumulating subvarieties. Hence, Conjecture 2.4.3 is only formulated on a Zariski open set U of X. The constant $C_{H_{\omega_X^{-1}}}(X)$ has been given a conjectural interpretation by Peyre [65] discussed in Section 2.4.1.

In the case when K is a global field of positive characteristic, Peyre [67, Theorem 3.5.1] proposed an analogue of [65]. Given a Fano variety X defined over a function field K, we are interested in the behaviour of

$$N_X(M) := \#\left\{P \in X(K) : H_{\omega_X^{-1}}(P) = q^M\right\},$$

as $M \to \infty$. Then, using the anticanonical height zeta function of X given by

$$Z_{H_{\omega_X^{-1}}}(s) = \sum_{x \in X(K)} \frac{1}{H_{\omega_X^{-1}}(x)^s}.$$

and applying the Wiener–Ikehara Theorem [70, Theorem 17.4], we obtain

$$N_X(M) \sim c^*_{H_{\omega_X^{-1}}}(X) \frac{(\log q)^{r_X}}{(r_X - 1)!} q^M M^{r_X - 1}, \tag{2.7}$$

as $M \to \infty$, where

$$c^*_{H_{\omega_X^{-1}}}(X) = \lim_{s \to 1} (s-1)^{r_X} Z_{H_{\omega_X^{-1}}}(s). \tag{2.8}$$

Manin's conjecture predicts that the constant agrees with the prediction of Peyre as defined in [9, 10, 67] and that the rank r_X of the Picard group of X is equal to the the multiplicity of the pole of the anticanonical height zeta function $Z_{H_{\omega_X^{-1}}}(s)$ of X at $s = 1$.

2.4.1 *Peyre's constant*

In this section, we will discuss the leading constant in Manin's conjecture and its geometric interpretation as given by Peyre which can be found in [65, 9, 10, 67]. Let X be a Fano variety defined over a global field K. Let S a finite subset of the set of places Ω_K

of K containing all ramified and infinite places. Then, the Peyre constant with respect to the anticanonical height $\mathcal{H} := H_{\omega_X^{-1}}$ is given by

$$c^*_{\mathcal{H}}(X) = \alpha^*(X)\beta(X)\tau_{\mathcal{H}}(X). \tag{2.9}$$

and $c_{\mathcal{H}}(X) = \alpha(X)\beta(X)\tau_{\mathcal{H}}(X)$, where $\alpha(X) = \frac{\alpha^*(X)}{(r_X-1)!}$ and r_X is the rank of the Picard group of X. The first two factors are geometric invariants and are independent of the height. We have

$$\alpha^*(X) = \int_{C^\vee_{\text{eff}}(X)} e^{-\langle \omega_X^{-1}, y\rangle} \, \mathrm{d}y, \tag{2.10}$$

where C_{eff} is the effective cone of X as defined in Section 2.4,

$$C^\vee_{\text{eff}}(X) = \{y \in \mathrm{Pic}(X)^\vee_{\mathbb{R}} \mid \langle x, y\rangle \geqslant 0 \text{ for all } x \in C_{\text{eff}}(X)\} \tag{2.11}$$

is the dual of the effective cone and $\mathrm{d}y$ is the normalised Lebesgue measure on $\mathrm{Pic}(X)^\vee$. The second invariant is

$$\beta(X) = \# H^1(K, \mathrm{Pic}(X^s)), \tag{2.12}$$

where X^s is the separable closure of V. The third factor is given by

$$\tau_{\mathcal{H}}(X) = \omega_{\mathcal{H}}(\overline{X(K)}) = C_{K,X} l_S(X) \prod_{v \in S} \omega_{X,v} \prod_{v \notin S} \lambda_v^{-1} \omega_{X,v}, \tag{2.13}$$

where $\lambda_v = L_v(1, \mathrm{Pic}(X^s))$ and

$$C_{K,X} = \begin{cases} \Delta_K^{-\dim(X)/2}, & \text{if } K \text{ is a number field,} \\ q^{(1-g_K)\dim(X)}, & \text{if } K \text{ is a function field,} \end{cases}$$

and

$$l_S(X) = \lim_{s \to 1} (s-1)^{\mathrm{rk}(\mathrm{Pic}(X^s))} \prod_{v \notin S} L_v(s, \mathrm{Pic}(X^s)).$$

From now on, let K be a function field of degree e over $\mathbb{F}_q(t)$. Given a place $v \in \Omega_K$, let κ_v be the residue field with respect to v. If $v \notin S$ and $\mathcal{X}$ is a smooth projective model of X, then

$$\omega_{X,v} = \frac{\#\mathcal{X}(\kappa_v)}{(\#\kappa_v)^{\dim(X)}}.$$

Example 2.4.4. As in [58, Example 1.31], in the case when $X = \mathbb{P}^n$ is defined over K, we have $\mathrm{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$ and $\omega_{\mathbb{P}^n}^{-1} = \mathcal{O}_{\mathbb{P}^n}(n+1)$. Thus, the first two factors are

$$\alpha^*(\mathbb{P}^n) = \frac{1}{n+1} \quad \text{and} \quad \beta(\mathbb{P}^n) = 1.$$

Moreover, we can take $S = \varnothing$. We have $\#\kappa_v = q_v$ and

$$\prod_{v \notin S} L_v(s, \mathrm{Pic}(X^s)) = \prod_{v \in \Omega_K} \frac{1}{1 - q_v^{-s}} = \zeta_K(s).$$

Hence, we obtain that $l_S(\mathbb{P}^n)$ is equal to

$$\lim_{s \to 1}(s-1) L_K(q^{-s}) \zeta_{\mathbb{F}_q(t)}(s) = L_K(q^{-1}) \lim_{s \to 1}(s-1)\zeta_{\mathbb{F}_q(t)}(s) = \frac{J_K q^{1-g_K}}{(q-1)\log q}, \tag{2.14}$$

where L_K is called the L-polynomial of $K/\mathbb{F}_q$ and its value at q^{-1} follows from [77, Theorem 5.1.15], g_K is the genus of K and J_K is the number of divisor classes of degree 0 (cardinality of the Jacobian) of K. Moreover,

$$\omega_{\mathbb{P}^n, v} = \frac{(q_v^{n+1} - 1)/(q_v - 1)}{q_v^n} = \frac{1 - q_v^{1-n}}{1 - q_v^{-1}},$$

and the convergence factors are given by $\lambda_v^{-1} = L_v(1, \mathrm{Pic}(V^s))^{-1} = 1 - q_v^{-1}$. Then

$$\prod_{v \in \Omega_K} \lambda_v^{-1} \omega_v(X(K_v)) = \prod_{v \in \Omega_K} \left(1 - q_v^{-1-n}\right) = \zeta_K(n+1)^{-1}.$$

Putting all this together, we obtain

$$c_{\mathcal{H}}(\mathbb{P}^n) = \frac{1}{\log q^{n+1}} \left(\frac{1}{q^{g_K - 1}}\right)^{n+1} \frac{J_K}{(q-1)\zeta_K(n+1)}.$$

The first counterexample to Conjecture 2.4.3 was provided by Batyrev and Tschinkel [4] who considered the hypersurface $X \in \mathbb{P}^3 \times_{\mathbb{Q}} \mathbb{P}^3$ defined by $\sum_{i=0}^3 x_i y_i^3 = 0$. In particular, they prove that any dense open set of X contains too many rational points over a number field K containing a primitive third root of unity. Recent work of Loughran [59] and Frei–Loughran–Sofos [28] generalises this; thus, provides counterexamples for any number field K. This led to a refined version of Manin's conjecture in which one is allowed to remove a finite number of thin sets as defined by Serre [75, §3.1]. We recall this definition below.

Definition 2.4.5. Let V be an irreducible algebraic variety defined over a field K. A *thin* set T is a set that is contained in a finite union of sets of the type (C_1) or (C_2), where a subset $A \subseteq V(K)$ is of type (C_1) if there exists a proper closed subset W of V such that $A \subset W(K)$, and of type (C_2) if there exists an irreducible variety W of V of dimension equal to $\dim V$ and a generically surjective morphism $\pi : W \to V$ of degree $\geqslant 2$ with $A \subset \pi(W(K))$.

This version of the conjecture is supported by results of Le Rudulier [58, Theorem 4.2], who studied Hilbert schemes of points, and Browning–Heath-Brown [14, Theorem 1.1], who studied the hypersurface $X \in \mathbb{P}^3 \times_{\mathbb{Q}} \mathbb{P}^3$ defined by $\sum_{i=0}^{3} x_i y_i^2 = 0$. In Chapter 4 we show that by eliminating an exceptional thin set, the refined version of Manin's conjecture holds for an example which can be seen as a generalisation of the function field analogue of Le Rudulier's result. This is the first result towards the thin set version of Manin's conjecture over function fields. Both the proof in [58] and the proof of the main result in Chapter 4 rely on previous work on counting projective points in fields extensions. Thus, we will focus on this topic in the following section.

2.5 COUNTING POINTS IN FIELD EXTENSIONS

As described in Chapter 1, the distribution of points of bounded height on algebraic varieties is a major question in arithmetic geometry. One of the simplest varieties to be studied is the projective space $\mathbb{P}^n$ over a global field K of degree $e \geqslant 1$. Given a point $x = [x_0 : \ldots : x_n]$ in $\mathbb{P}^n(\overline{K})$, define $K(x)$ to be the field obtained by adjoining all quotients x_i/x_j. Then, the *degree* of x is equal to $[K(x) : K]$. As in Section 2.3, let Ω_K be the set of places of K. Recall that a global field K satisfies the product formula in Lemma 2.3.1, i.e. for any $x \in K^\times$, we have

$$\prod_{v \in \Omega_K} \|x\|_v = 1.$$

Suppose K is a number field. Then $e = [K : \mathbb{Q}]$. If x is a point of degree $d \geqslant 1$ over K with homogeneous coordinates $[x_0 : \ldots : x_n]$ we define the *absolute multiplicative height* of x to be

$$H(x) = \left(\prod_{v \in \Omega_{K(x)}} \|x\|_v \right)^{\frac{1}{d}}. \tag{2.15}$$

Then, given $n \in \mathbb{Z}_{>1}$, $d \in \mathbb{Z}_{\geqslant 1}$, $B \in \mathbb{R}_{>0}$ we define

$$N_K(n, d, B) = \#\left\{x \in \mathbb{P}^{n-1}(\overline{\mathbb{Q}}) : H(x) \leqslant B, [K(x) : K] \leqslant d\right\}. \tag{2.16}$$

Then, by Northcott's Theorem, $N_K(n, d, B)$ is bounded. Moreover, there are classical results which describe $N_{\mathbb{Q}}(n, 1, B)$ as the number of primitive lattice points in a ball and give asymptotics as $B \to \infty$. The first generalisation for the cases where $e > 1$ was given by Schanuel [71] who proved that

$$N_K(n, 1, B) = S_K(n, 1)B^{ne} + O_{k,n}(B^{ne-1}),$$

where a $\log B$ term must be introduced in the error term if $K = \mathbb{Q}$ and $n = 2$. The leading constant is called the *Schanuel constant*

$$S_K(n+1, 1) = \left(\frac{2^r(2\pi)^s}{|\Delta|^{1/2}}\right)^{n+1} \frac{(n+1)^{r+s-1}hR}{w\zeta_K(n+1)}, \tag{2.17}$$

where r is the number of real embeddings of K, s the number of pairs of distinct complex conjugate embeddings of K, Δ the discriminant of K, h is the class number of K, R the regulator of K, w the number of roots of unity in K, and ζ_K the Dedekind zeta-function of K. A further generalisation was given by Schmidt [73] for quadratic number fields, and for $d > 2$ [72] he only gives upper and lower bounds for $N_K(n, d, B)$. Of particular interest to us is the case when $n = 3$, $e = 1$, and $d = 2$ which states that

$$N_{\mathbb{Q}}(3, 2, B) = \frac{24 + 2\pi^2}{\zeta(3)^2} B^6 \log B + O\left(B^6\sqrt{\log B}\right), \tag{2.18}$$

where the leading constant is a sum of Schanuel constants over extension fields K of degree $d = 2$ over $\mathbb{Q}$,

$$S_{\mathbb{Q}}(3, 2) = \sum_{[K:\mathbb{Q}]=2} S_K(3, 1) = S_K^+(3, 1) + S_K^-(3, 1),$$

where $\pm$ denotes whether the discriminant $\Delta_K > 0$ or < 0, respectively. In his thesis, Gao generalised Schmidt's results and computed asymptotics for $N_{\mathbb{Q}}(n, d, B)$ as $B \to \infty$ provided that $n > d + 1 > 3$. The case when $n = 2$ has been treated by Masser-Vaaler [62, 63] who provided asymptotics for $N_K(2, d, B)$ as $B \to \infty$. A further improvement has been made by Widmer [89] who showed that for $n > \frac{5d}{2} + 3 + \frac{2}{ed}$, we have

$$N_K(n, d, B) = S_K(n, d)B^{ne} + O(B^{ne-1/d}),$$

where $S_K(n, d)$ is the sum of Schanuel constants $S_K(n, 1)$ over extension fields K of degree d over K.

Now, suppose that K is a degree e extension of $\mathbb{F}_q(t)$. Note that by [1, Chapter 15.1], if x is a point of degree d in K, the effective degree of $K(x)$ over K is d and the effective degree of $K(x)$ over $\mathbb{F}_q(t)$ is ed. Thus, the height of x must be of the form $q^{M/ed}$, for some $M \in \mathbb{Z}_{\geqslant 0}$, which suggests we should count points of fixed height instead. If $[x_0 : \ldots : x_n]$

are the homogeneous coordinates of x, we define the *absolute multiplicative height* of x to be

$$H(x) = \left(\prod_{v \in \Omega_{K(x)}} \|x\|_v \right)^{\frac{1}{de}}, \tag{2.19}$$

since we can regard x as a point over $\mathbb{F}_q(t)$. Let $n \in \mathbb{Z}_{>1}$, $d \in \mathbb{Z}_{\geqslant 1}$, $M \in \mathbb{Z}_{\geqslant 0}$ and define the counting function analogous to (2.16) to be

$$N_K(n,d,M) = \#\left\{ x \in \mathbb{P}^{n-1}(\overline{\mathbb{F}_p(t)}) : H(x) = q^{\frac{M}{ed}}, [K(x):K] = d \right\}. \tag{2.20}$$

By an analogue of Northocott's theorem, the quantity $N_K(n,d,M)$ is bounded. Wan [86] and DiPippo [21] provided the function field correspondent of [71] which was originally stated without proof by Serre [74, Section 2.5]. The analogue of Schanuel's constant is given by

$$S_K(n,1) = \left(\frac{1}{q^{(g_K - 1)}} \right)^n \frac{J_K}{(q-1)\zeta_K(n)}, \tag{2.21}$$

where g_K is the genus of K, J_K is the number of divisor classes of degree 0 (cardinality of the Jacobian) of K, and ζ_K is the zeta function of K. Further work has been done by Thunder [82], who provided analogues of [62, 63] and relates the count to the geometry of Schubert varieties. Later, Thunder and Widmer [83, Theorem 2] give a more robust analogue of [71] and prove that if $M \geqslant 2g_K - 1$ and $1/4 \geqslant \varepsilon > 0$, then for all integers $n \geqslant 4$ we have

$$N_K(n,1,M) = S_K(n,1)q^{Mn} + O\left(q^{M(1+\varepsilon)} q^{g_K(n-2-2\varepsilon)}\right),$$

and for $n = 2, 3$, we have

$$N_K(n,1,m) = S_K(n,1)q^{Mn} + O\left(q^{M(1+\varepsilon)} q^{g_K(1+\varepsilon)}\right).$$

Moreover, if $M < 2g_K - 1$, then for all $\varepsilon > 0$ and all integers $n \geqslant 2$ we have

$$N_K(n,1,m) \ll q^{M((n+1)/2+\varepsilon)}.$$

We remark that all the implicit constants depend only on n, e, q and ε. Another result [83, Theorem 1] can be seen as an analogue of Widmer [89]. More precisely, they obtain that for all integers n and $d > 1$ and $\varepsilon > 0$ with $n > 2d + 3 + \varepsilon$, then for $M \in \mathbb{Z}_{\geqslant 0}$ we have

$$N_K(n,d,M) = S_K(n,d)q^{Mn} + O\left(q^{\frac{M}{2}(n+2d+3+\varepsilon)}\right),$$

where the implicit constant depends only on K, n, d and ε and the sum

$$S_K(n,d) = \sum_{[L:K]=d} S_L(n,1)$$

converges for all integers n and $d > 1$ satisfying $n > d+2$. Kettlestrings and Thunder [46] provide an improvement of [83] in the case of quadratic extensions. The proof of our main result in Chapter 4 relies on their result for $d=2$ and $n=3$. This can be seen as a function field analogue of [73], but generalised to extensions of any degree $e \geqslant 1$. More precisely, we have

$$N_K(3,2,m) = 2Mq^{3M}\left(S_K(3,1)\right)^2 + O(\sqrt{M}q^{3M})$$

where the implicit constant depends only on K.

2.5.1 *The choice of height*

So far we have introduced various choices of height functions and in this section we will clarify through an example why using an absolute height in the case of function fields is more natural from a counting point of view compared to using a height associated to the anticanonical divisor of the variety. We will consider $V = \mathbb{P}^n$ defined over a global function field K.

Denote by H_n the usual absolute height on projective space $\mathbb{P}^n$ with respect to the ground field $\mathbb{F}_q(t)$. More precisely, given a point $x \in \mathbb{P}^n(\overline{\mathbb{F}_q(t)})$ of degree d over K with homogeneous coordinates $[x_0 : \ldots : x_n]$ we have

$$H_n(x) = \left(\prod_{v\in\Omega_{K(P)}} \max_{0\leqslant i\leqslant n} |x_i|_v\right)^{\frac{1}{de}}. \tag{2.22}$$

Regarding $\mathbb{P}^n$ over the ground field K, we have that for $x \in \mathbb{P}^n(\overline{K})$, the absolute height on projective space associated to the anticanonical line bundle is

$$H_{\omega_{\mathbb{P}^n}^{-1}}(x) = H_n(x)^{(n+1)e}.$$

Counting points with the usual height H_n we note that

$$\#\left\{P \in \mathbb{P}^n(\mathbb{F}_q(t)) : H_n(P) \leqslant q^M\right\}$$

can be rewritten as

$$\frac{1}{q-1}\#\left\{\mathbf{x} \in \mathbb{F}_q[t]^{n+1}\backslash\{\mathbf{0}\} : x_i \text{ coprime}, |x_i| \leqslant q^M \text{ for all } i\right\}. \tag{2.23}$$

We can use the Möbius function to detect the coprimality condition via Möbius inversion as follows. By Lemma 2.2.2, we can rewrite the above as

$$\frac{1}{q-1}\sum_{\substack{r\in\mathbb{F}_q[t]\\ \text{monic}}}\#\left\{\mathbf{x}\in\mathbb{F}_q[t]^{n+1}\backslash\{\mathbf{0}\}:\gcd(x_0,\ldots,x_n)=r,|x_i|\leqslant q^M\text{ for all }i\right\}\sum_{\substack{k\in\mathbb{F}_q[t]\\ \text{monic}\\ k|r}}\mu(k).$$

Now, we can replace the sum over k by

$$\sum_{\substack{k\in\mathbb{F}_q[t]\\ \text{monic}}}\mu(k)\sum_{\substack{g\in\mathbb{F}_q[t]\\ \text{monic}}}\mathbb{1}_{g=\frac{r}{k}}$$

and, after changing the order of summation, (2.23) becomes

$$\frac{1}{q-1}\sum_{\substack{k\in\mathbb{F}_q[t]\\ \text{monic}}}\mu(k)\sum_{\substack{g\in\mathbb{F}_q[t]\\ \text{monic}}}\#\left\{\mathbf{x}\in\mathbb{F}_q[t]^{n+1}\backslash\{\mathbf{0}\}:\gcd(x_0,\ldots,x_n)=kg,|x_i|\leqslant q^M\forall i\right\}.$$

After a change of variables, we obtain

$$\begin{aligned}&\frac{1}{q-1}\sum_{\substack{k\in\mathbb{F}_q[t]\\ \text{monic}}}\mu(k)\sum_{\substack{g\in\mathbb{F}_q[t]\\ \text{monic}}}\#\left\{\mathbf{y}\in\mathbb{F}_q[t]^{n+1}\backslash\{\mathbf{0}\}:\gcd(y_0,\ldots,y_n)=g,|y_i|\leqslant\frac{q^M}{|k|}\forall i\right\}\\ =&\frac{1}{q-1}\sum_{\substack{k\in\mathbb{F}_q[t]\\ \text{monic}}}\mu(k)\#\left\{\mathbf{y}\in\mathbb{F}_q[t]^{n+1}\backslash\{\mathbf{0}\}:|y_i|\leqslant\frac{q^M}{|k|}\text{ for all }i\right\}.\end{aligned}$$

Hence, (2.23) is equal to

$$\begin{aligned}&\frac{1}{q-1}\sum_{j\geqslant 0}\sum_{\substack{k\in\mathbb{F}_q[t]\\ \text{monic}\\ |k|=q^j}}\mu(k)\#\left\{\mathbf{y}\in\mathbb{F}_q[t]^{n+1}\backslash\{\mathbf{0}\}:|y_i|\leqslant q^{M-j}\text{ for all }i\right\}\\ =&\frac{1}{q-1}\sum_{j\geqslant 0}\sum_{\substack{k\in\mathbb{F}_q[t]\\ \text{monic}\\ |k|=q^j}}\mu(k)\left(q^{(n+1)(M+1-j)}-1\right).\end{aligned}$$

Finally, by (2.2), we obtain

$$=\frac{q^{(n+1)(M+1)}\left(1-q^{1-(n+1)}\right)}{q-1}+1.$$

Therefore, writing $\#\left\{P\in\mathbb{P}^n(\mathbb{F}_q(t)):H_n(P)=q^M\right\}$ as

$$\#\left\{P\in\mathbb{P}^n(\mathbb{F}_q(t)):H_n(P)\leqslant q^M\right\}-\#\left\{P\in\mathbb{P}^n(\mathbb{F}_q(t)):H_n(P)\leqslant q^{M-1}\right\}$$

we get

$$\frac{q^{n+1}}{(q-1)\zeta_{\mathbb{F}_q(t)}(n+1)}q^{(n+1)M},$$

where

$$\zeta_{\mathbb{F}_q(t)}(s) = \left(1-q^{-s}\right)^{-1}\left(1-q^{1-s}\right)^{-1}.$$

Thus,

$$\#\left\{P \in \mathbb{P}^n(\mathbb{F}_q(t)) : H_n(P) = q^M\right\} = S_{\mathbb{F}_q(t)}(n+1,1)q^{(n+1)M}. \tag{2.24}$$

We will now show how this relates to the height zeta function corresponding to H_n which is defined as

$$Z_n(s) = \sum_{\mathbf{x}\in\mathbb{P}^n(\mathbb{F}_q(t))} \frac{1}{H_n(\mathbf{x})^s}.$$

To compute it, we rewrite it as

$$\frac{1}{q-1}\sum_{j\geqslant 0}\sum_{\substack{k\in\mathbb{F}_q[t]\\ k \text{ monic}\\ |k|=q^j}} \mu(k)\sum_{N\geqslant 0}\frac{1}{q^{Ns}}\#\left\{\mathbf{x}\in\mathbb{F}_q[t]^{n+1}\backslash\{\mathbf{0}\} : |\mathbf{x}| = q^{N-j}\right\}.$$

Recall (2.2). If $N=0$, then j must be 0 and

$$\#\left\{\mathbf{x}\in\mathbb{F}_q[t]^{n+1}\backslash\{\mathbf{0}\} : |\mathbf{x}| = 1\right\} = q^{n+1}-1.$$

If $N \geqslant 1$, then $\#\left\{\mathbf{x}\in\mathbb{F}_q[t]^{n+1}\backslash\{\mathbf{0}\} : x_i \text{ coprime}, |\mathbf{x}| = q^{N-j}\right\}$ is equal to

$$q^{(N-j)(n+1)}\left(q^{n+1}-1\right).$$

Hence, we have

$$Z_n(s) = \frac{q^{n+1}-1}{q-1} + \frac{q^{n+1}-1}{q-1}\sum_{N\geqslant 1}\frac{q^{N(n+1)}-q^{(N-1)(n+1)+1}}{q^{Ns}} = \frac{q^{n+1}-1}{q-1}\cdot\frac{1-q^{1-s}}{1-q^{n+1-s}}.$$

This has a simple pole at $s = n+1$ with residue

$$\lim_{s\to n+1}(s-n-1)Z_n(s) = \frac{S_{\mathbb{F}_q(t)}(n+1,1)}{\log q}.$$

We would like to apply the function field version of the Wiener-Ikehara Tauberian Theorem as found in [70, Theorem 17.1]. Since, $Z_n(s)$ has a pole at $s = n+1$, the result does not apply immediately. However, substituting s by $s+n$ we get

$$Z_n(s+n) = \frac{q^{n+1}(1-q^{-n-1})}{q-1}\cdot\frac{1-q^{1-n-s}}{1-q^{1-s}}$$

has a simple pole at $s = 1$ with the same residue. Using the power series expansion in $q^{-s} = u$ of $Z_n(s+n)$, we can now apply the theorem to obtain

$$q^{-nN}\#\left\{\mathbf{x}\in\mathbb{P}^n(\mathbb{F}_q(t)) : H_n(\mathbf{x}) = q^M\right\} = \frac{q^{n+1}(1-q^{-n-1})(1-q^{-n})}{(q-1)\log q}\log(q)q^M,$$

and thus,

$$\#\left\{\mathbf{x} \in \mathbb{P}^n(\mathbb{F}_q(t)) : H(\mathbf{x}) = q^M\right\} = S_{\mathbb{F}_q(t)}(n+1,1)q^{M(n+1)},$$

which agrees with the direct count (2.24).

However, Manin's conjecture and Peyre's constant are given using the anticanonical height. If we let $\mathcal{H}(x) = H_{\omega_{\mathbb{P}^n}^{-1}}(x)$, then following similar steps as above we obtain that $\#\left\{P \in \mathbb{P}^n(\mathbb{F}_q(t)) : \mathcal{H}(P) \leqslant q^M\right\}$ is equal to

$$\begin{aligned}&\frac{1}{q-1}\#\left\{\mathbf{x} \in \mathbb{F}_q[t]^{n+1}\backslash\{\mathbf{0}\} : x_i \text{ coprime}, |x_i|^{n+1} \leqslant q^M \text{ for all } 1 \leqslant i \leqslant n+1\right\}\\ =&\frac{1}{q-1}\sum_{j\geqslant 0}\sum_{\substack{k\in\mathbb{F}_q[t]\\ \text{monic}\\ |k|=q^j}}\mu(k)\left(q^{(n+1)\left(\left\lfloor\frac{M}{n+1}\right\rfloor-j+1\right)}-1\right).\end{aligned}$$

Now by (2.2), we have that this equals

$$1+\frac{q^{n+1}-q}{q-1}q^{(n+1)\left\lfloor\frac{M}{n+1}\right\rfloor}.$$

Then, $\#\left\{P \in \mathbb{P}^n(\mathbb{F}_q(t)) : \mathcal{H}(P) = q^M\right\}$ is given by

$$\begin{aligned}&\frac{q^{n+1}-q}{q-1}\left(q^{(n+1)\left\lfloor\frac{M}{n+1}\right\rfloor}-q^{(n+1)\left\lfloor\frac{M-1}{n+1}\right\rfloor}\right)\\ =&\frac{q^{n+1}-q}{q-1}\cdot\begin{cases}0, & \text{if } M \not\equiv 0 \bmod n+1,\\ q^M\left(1-q^{-(n+1)}\right), & \text{if } M \equiv 0 \bmod n+1,\end{cases}\end{aligned}$$

which makes using the anticanonical height is not as natural. However, taking an average, we obtain

$$\frac{S_{\mathbb{F}_q(t)}(n+1,1)}{n+1}q^M. \tag{2.25}$$

The height zeta function corresponding to $\mathcal{H}$ is given by

$$\mathcal{Z}(s) = \sum_{\mathbf{x}\in\mathbb{P}^n(\mathbb{F}_q(t))}\frac{1}{\mathcal{H}(\mathbf{x})^s} = \sum_{\mathbf{x}\in\mathbb{P}^n(\mathbb{F}_q(t))}\frac{1}{H_n(\mathbf{x})^{(n+1)s}} = Z_n((n+1)s).$$

This has a simple pole at $s=1$ with residue

$$\frac{S_{\mathbb{F}_q(t)}(n+1,1)}{(n+1)\log q}.$$

Applying the function field version of the Wiener-Ikehara Tauberian Theorem as found in [70, Theorem 17.1], we get

$$\#\left\{\mathbf{x} \in \mathbb{P}^n(\mathbb{F}_q(t)) : H(\mathbf{x}) = q^M\right\} = \frac{S_{\mathbb{F}_q(t)}(n+1,1)}{n+1}q^M,$$

which agrees with (2.25), and the constant matches Peyre's constant as computed in Example 2.4.4.

Since it is not difficult to relate the height zeta functions for these two choices of heights, in Chapter 4, we will use a height coming from the usual height on projective space for the part related to counting, but we will interpret the final results in terms of the anticanonical height.

2.6 THE CIRCLE METHOD

In the case of varieties for which the number of variables is large compared to the degree of its defining polynomials, one of the possible ways used to tackle Manin's conjecture is the Hardy–Littlewood circle method. We refer the reader to [84, 85, 13, 12] for a more detailed survey of the history and more recent developments of this method and its applications. A quick summary is also given below.

The circle method was originally designed by Hardy and Ramanujan [37] to solve the partition problem, that is to give an asymptotic formula for the number of ways in which a positive integer n can be written as a sum of positive integers. In a series of papers starting with [34, 35], Hardy and Littlewood have further developed the method and used it to give effective bounds for Waring's problem, which is a generalisation of the partition problem. This question originated in the eighteenth century when Waring [87] stated without proof that every natural number is a sum of at most 9 positive integral cubes and a sum of at most 19 fourth powers, and so forth. We remark that the proof that a number can be written as a sum of at most 4 squares is due to Lagrange and it appeared during the same year as Waring's paper, however, the statement had previously been mentioned in the literature. The fact that there exists a number s such that every positive integer n can be written as a sum of at most s k-th powers of positive integers was proved by Hilbert [41] through a combinatorial argument, however, his result is far from giving the smallest such s. A related question is to find the smallest number $G(k)$ such that every sufficiently large number is the sum of at most $G(k)$ k-th powers. Hardy and Littlewood [36] obtained the first result towards answering this in the 1920s, in particular $G(k) \leqslant (k-2)2^{k-1}+5$. We will give a summary of the ideas that lay at the basis of the original method, but we remark that there have been several refinements that led to significant improvements of the bound.

Let $r(n)$ denote the number of representations of a positive integer n as a sum of exactly s k-th powers of positive integers, i.e.

$$r(n) = \#\left\{(x_1, \ldots, x_s) \in \mathbb{N}^s | x_1^k + \ldots + x_s^k = n\right\}.$$

If $r(n) > 0$ for all large n, then we get that $G(k) \leqslant s$. Thus, obtaining an asymptotic formula for $r(n)$ as $n \to \infty$, we get a bound for $G(k)$. Introducing the truncated generating function

$$S(\alpha) = \sum_{1 \leqslant x \leqslant n^{1/k}} e\left(\alpha x^k\right),$$

we obtain that $S(\alpha)^s = \sum_{1 \leqslant x \leqslant n^s} r\left(x; n^{1/k}\right) e\left(\alpha x\right)$, where $r(n; N)$ counts the number of representations of n as a sum of s k-th powers of integers in $[1, N]$. On noting that $r(n) = r(n; n^{1/k})$ and recalling the orthogonality relation (2.1), we have reduced the problem to evaluating

$$r(n) = \int_0^1 S(\alpha)^s e(-\alpha n)\, \mathrm{d}\alpha.$$

The idea is to break the integral into two parts: the *major arcs* $\mathfrak{M}$ where the function $S(\alpha)$ is large and which give a significant contribution, and the *minor arcs* $\mathfrak{m}$ from which we get a small contribution. This is based on work of Weyl [88] who observed that when α is close to rationals $\frac{a}{q}$, where a, q are coprime and q is "small", then $S(\alpha)$ has a peak. By Dirichlet's approximation Theorem, every α can be approximated in this way, that is, for every real α and Q, there exist integers a, q with $1 \leqslant q \leqslant Q$ and $|q\alpha - a| \leqslant Q^{-1}$. Then, we would like to choose the parameter Q such that all intervals centred at such α are disjoint. We define the major arcs to be the union of these intervals and the minor arcs to be $[0, 1] - \mathfrak{M}$. The next step is to estimate these two contributions.

Splitting the unit integral into major and minor arcs is more challenging if one tries to extend the method to number fields. However, Siegel [78, 79] completed this generalisation and obtained a bound for $G(k)$ depending on the degree of the number field. This dependance was later removed in work of Birch [7] and Ramanujan [69].

Another celebrated application of the circle method is due to Birch [8] in his study of complete intersections in which he obtains an asymptotic formula for the number of integer points in a box on a smooth variety $V \subseteq \mathbb{P}^n$ defined by r homogeneous polynomials $f_1, \ldots, f_r$ of degree d over $\mathbb{Q}$ with $n > r(r+1)(d-1)2^{d-1}$. Later on, noting that if $\dim V \geqslant 3$, then V is Fano as soon as $n \geqslant rd$ and $\operatorname{Pic} X \cong \operatorname{Pic} \mathbb{P}^n \cong \mathbb{Z}$ has rank 1, Franke, Manin and Tschinkel [27] showed that this result and Manin's conjecture (see

Conjecture 2.4.3) for complete intersections of large dimension are compatible. We will present their idea below. Noting that $\omega_V^{-1} = \mathcal{O}_V(n+1-rd)$ and that $-\mathbf{x}$ and $\mathbf{x}$ represent the same point in $\mathbb{P}^n$, we have that

$$\#\left\{x \in V(\mathbb{Q}) : H_{\omega_V^{-1}}(x) \leqslant B\right\}$$

is the same as

$$\frac{1}{2}\#\left\{\mathbf{x} \in \mathbb{Z}^{n+1}\backslash\{\mathbf{0}\} : (x_0,\dots,x_n)=1, f_1(\mathbf{x})=\dots=f_r(\mathbf{x})=0, \max_{0\leqslant i\leqslant n}|x_i|^{n+1-rd} \leqslant B\right\}.$$

Then, using Möbius inversion to remove the coprimality condition, we get that the above is equal to

$$\frac{1}{2}\sum_k \mu(k)\#\left\{\mathbf{x} \in k\mathbb{Z}^{n+1}\backslash\{\mathbf{0}\} : f_1(\mathbf{x})=\dots=f_r(\mathbf{x})=0, \max_{0\leqslant i\leqslant n}|x_i|^{n+1-rd} \leqslant B\right\},$$

which by Birch's result, is

$$\sim \frac{c}{2}\sum_k \mu(k)\frac{B}{k^{n+1-rd}} = \frac{c}{2\zeta(n+1-rd)}B,$$

as $B \to \infty$. This agrees with Manin's prediction. Moreover, the description of the leading constant in [8, Theorem 1] given in terms of the singular series and singular integral that turn up in the circle method provided motivation for the geometric interpretation of Peyre [65] (see Section 2.4.1).

The work of Birch had a strong influence on number theory and the circle method has been generalised to arbitrary number fields by Skinner [80] and the function field $\mathbb{F}_q(t)$ by Lee [55, 56]. The latter has been further refined by Browning–Vishe [15] and it is the method we use to obtain the results concerning cubic hypersurfaces in Chapter 3.

As described in [15], applying the circle method over the function field $\mathbb{F}_q(t)$ has several advantages. In particular, since all the absolute values of $\mathbb{F}_q(t)$ are non-archimedean, dissecting the analogue of the unit interval into non-overlapping arcs is a much simpler process. All that is required is an analogue of Dirichlet's approximation Theorem [55, Lemma 5.1]. Thus, there is no division into major and minor arcs. Moreover, in the case of cubic hypersurfaces [15], this leads immediately to the possibility of using a double Kloosterman refinement, that is, non-trivial averaging both over the numerator and denominator of the fractions $\frac{a}{q}$ occurring in the exponential sums.

3

RATIONAL CURVES ON CUBIC HYPERSURFACES

This chapter is taken from the author's previous work [64].

3.1 INTRODUCTION

Let $k = \mathbb{F}_q$ be a finite field and let $F \in k[x_1, \ldots, x_n]$ denote a non-singular homogeneous polynomial of degree 3. Moreover, let $X \subset \mathbb{P}_k^{n-1}$ be the smooth cubic hypersurface defined over k by $F = 0$. Let C be a smooth projective curve over k. We recall that a C_m*-field* is a field for which any homogeneous polynomial of degree d in n variables, with $d^m < n$ and $m \geqslant 1$, has a non-trivial zero. Moreover, C_1-fields are called *quasi-algebraically closed* and an example is given by finite fields, due to the Chevalley–Warning theorem. Then $k(C)$ has transcendence degree 1 over a C_1-field and, by the Lang–Tsen theorem [33, Theorem 3.6], the set $X(k(C))$ of $k(C)$-rational points on X is non-empty for $n \geqslant 10$. This still holds for X singular. In this paper we are interested in the case $C = \mathbb{P}^1$, writing $K = k(t)$ denote the function field of C over k.

A *degree d k-rational curve on X* is a non-constant morphism $f : \mathbb{P}_k^1 \to X$ given by

$$f = (f_1(u, v), \ldots, f_n(u, v)), \tag{3.1}$$

where $f_i \in k[u, v]$ are homogeneous polynomials of degree $d \geqslant 1$, with no non-constant common factor in $k[u, v]$, such that

$$F(f_1(u, v), \ldots, f_n(u, v)) \equiv 0.$$

Such a curve is said to be *m-pointed* if it is equipped with a choice of m distinct points $P_1, \ldots, P_m \in X(k)$ called the *marks* through which the curve passes. Two such rational curves f and f' are isomorphic is there exists an automorphism $\phi : \mathbb{P}_k^1 \to \mathbb{P}_k^1$ such that $f = f' \circ \phi$. Up to isomorphism, these curves are parametrised by the moduli space $\mathcal{M}_{0,m}(\mathbb{P}_k^1, X, d)$. There is a natural associated moduli problem, the Kontsevich moduli space of stable maps which gives a compactification $\overline{\mathcal{M}}_{0,m}(\mathbb{P}_k^1, X, d)$ of $\mathcal{M}_{0,m}(\mathbb{P}_k^1, X, d)$.

Suppose from now on that $\#k = q$ and $\mathrm{char}(k) > 3$. In [49, Example 7.6], Kollár proves that there exists a constant c_n depending only on n such that for any $q > c_n$ and any point $x \in X(k)$, there exists a k-rational curve of degree at most 216 on X passing through x. In our investigation, we focus on the case $m = 2$ of 2-pointed rational curves on X.

Let $\mathbf{H}$ denote the *Hessian matrix*

$$\mathbf{H}(\mathbf{x}) = \left(\frac{\partial^2 F}{\partial x_i \partial x_j}\right)_{1 \leqslant i,j \leqslant n}.$$

and $H = 0$, where $H(\mathbf{x}) = \det \mathbf{H}(\mathbf{x})$, the *Hessian hypersurface* associated to F. Now let $\mathbf{a}, \mathbf{b} \in \mathbb{F}_q^n \backslash \{\mathbf{0}\}$ be such that $F(\mathbf{a}) = F(\mathbf{b}) = 0$ and $H(\mathbf{b}) \neq 0$. The condition that one of the two fixed points is not on the Hessian comes from our analysis of certain oscillatory integrals (see Lemma 3.2.7) and is crucial to the proof. Write $a = [\mathbf{a}]$, $b = [\mathbf{b}]$ for the corresponding points in $X(k)$. As is well-known (see [42, Lemma 1], for example), the Hessian $H(\mathbf{x})$ does not vanish identically on X, since $\mathrm{char}(k) > 3$. The main goal of this paper is to obtain an asymptotic formula for the number of rational curves of degree d on X passing through a and b. Denote the space of such curves by $\mathrm{Mor}_{d,a,b}(\mathbb{P}^1_k, X)$. We remark that rational curves in $\mathrm{Mor}_{d,a,b}(\mathbb{P}^1_k, X)$ are not counted up to isomorphism, unlike in $\mathcal{M}_{0,2}(\mathbb{P}^1_k, X, d)$. Thus, since the dimension of the automorphism group of $\mathbb{P}^1$ is 3, we have $\dim \mathrm{Mor}_{d,a,b}(\mathbb{P}^1_k, X) - 3 = \dim \mathcal{M}_{0,2}(\mathbb{P}^1_k, X, d)$. We can write the f_i in (3.1) explicitly as

$$f_i(u,v) = \alpha_d^{(i)} u^d + \alpha_{d-1}^{(i)} u^{d-1} v + \ldots + \alpha_0^{(i)} v^d, \tag{3.2}$$

where $\alpha_j^{(i)} \in k$ for $0 \leqslant j \leqslant d$ and $1 \leqslant i \leqslant n$. Then, we capture the condition that the rational curve f passes through the points $\mathbf{a}$ and $\mathbf{b}$ by selecting

$$\begin{aligned} \left(\alpha_0^{(1)}, \ldots, \alpha_0^{(n)}\right) &= \mathbf{a}, \\ \left(\alpha_d^{(1)}, \ldots, \alpha_d^{(n)}\right) &= \mathbf{b}. \end{aligned} \tag{3.3}$$

There exists a correspondence between the rational curves on X of bounded degree and the K-points on X of bounded height. Define $N_{a,b}(d)$ to be the number of polynomials $f_1, \ldots, f_n \in \mathbb{F}_q[t]$ of degree at most d whose constant coefficients are given by $\mathbf{a}$ and whose leading coefficients are given by $\mathbf{b}$, such that $F(f_1, \ldots, f_n) = 0$. Thus, $N_{a,b}(d)$ counts the $\mathbb{F}_q$-points $(f_1, \ldots, f_n)$ on the affine cone of $\mathrm{Mor}_{d,a,b}(\mathbb{P}^1_k, X)$, where the condition that $f_1, \ldots, f_n$ have no common factor is dropped. Using a version of the Hardy–Littlewood circle method for the function field K developed by Lee [55, 56], and further by Browning–Vishe [15], we shall obtain the following result.

Theorem 3.1.1. *Fix $k = \mathbb{F}_q$ with $\mathrm{char}(k) > 3$. Fix a smooth cubic hypersurface $X \subset \mathbb{P}_k^{n-1}$, where $n \geqslant 10$. Let $a, b \in X(k)$, not both lying on the Hessian. Then, we have*

$$N_{a,b}(d) = q^{(d-1)n-(3d-1)} + O\left(q^{\frac{5(d+2)n}{6}-\frac{5(d+2)}{3}} + q^{\frac{(5d+8)n}{6}-\frac{3d}{2}-\frac{8}{3}} + q^{\frac{3(d+5)n}{4}-\frac{3d+7}{4}}\right),$$

where the implied constant in the error term depends only on d and X.

Although it would be possible to generalise Theorem 3.1.1 to handle rational curves passing through any generic finite set of points in $X(k)$ (see Section 3.2), the main motivation for considering rational curves through two fixed points comes from the notion of rational connectedness. In [61], Manin defined R-equivalence on the set of rational points of a variety in order to study the parametrisation of rational points on cubic surfaces. We say that two points $a, b \in X(k)$ are *directly R-equivalent* if there is a morphism $f : \mathbb{P}^1 \to X$ (defined over k) with $f(0,1) = a$ and $f(1,0) = b$; the generated equivalence relation is called *R-equivalence*. In [81], Swinnerton-Dyer proved that R-equivalence is trivial on smooth cubic surfaces over finite fields; that is, all k-points are R-equivalent. Next, the result was generalised for smooth cubic hypersurfaces $X \subset \mathbb{P}_k^{n-1}$, if $n \geqslant 6$ by Madore in [60], and if $n \geqslant 4$ and $q \geqslant 11$ by Kollár in [49]. Moreover, Madore's result holds for X defined over any C_1 field. The study of R-equivalence is closely related to understanding the geometry of the moduli space of rational curves. In particular, it is interesting to study R-equivalence in the case of varieties with many rational curves. Such varieties are called *rationally connected* and were first studied by Kollár, Miyaoka and Mori in [50], and independently by Campana in [17]. Roughly speaking, Y is rationally connected if for two general points of Y there is a rational curve on Y passing through them. Thus, rationally connected varieties are varieties for which R-equivalence becomes trivial when one extends the ground field to an arbitrary algebraically closed field. Note that in the case of fields of positive characteristic one should consider *separably rationally connected* varieties. For precise definitions and a thorough introduction to the theory see Kollár [47], [48], and Kollár–Szabó [51].

Comparing the exponents in the main term and error terms in Theorem 3.1.1, we obtain

$$\begin{cases} (d-1)n-(3d-1) & \geqslant \frac{5(d+2)n}{6} - \frac{5d+10}{3} \\ (d-1)n-(3d-1) & \geqslant \frac{(5d+8)n}{6} - \frac{3d}{2} - \frac{8}{3} \\ (d-1)n-(3d-1) & \geqslant \frac{3(d+5)n}{4} - \frac{3d+7}{4} \end{cases} \iff \begin{cases} d & \geqslant \frac{16n-26}{n-8} \\ d & \geqslant \frac{14n-22}{n-9} \\ d & \geqslant \frac{19n-11}{n-9}, \end{cases}$$

since we consider $n \geqslant 10$. Moreover, for such n, we have $\frac{14n-22}{n-9} < \frac{19n-11}{n-9}$ and $\frac{16n-26}{n-8} < \frac{19n-11}{n-9}$, which leads to the following result.

Corollary 3.1.2. *Fix $k = \mathbb{F}_q$ with $\mathrm{char}(k) > 3$. Fix a smooth cubic hypersurface $X \subset \mathbb{P}_k^{n-1}$, where $n \geqslant 10$. Then there exists a constant $c_X > 0$ such that for any points $a, b \in X(k)$, not both on the Hessian hypersurface, and any $d \geqslant \frac{19n-11}{n-9}$, if $q \geqslant c_X$, then there exists an $\mathbb{F}_q$-rational curve $C \subset X$ of degree d that passes through a and b.*

This can also be deduced from a result of Pirutka [68, Proposition 4.3] which states that any two points $a, b \in X(k)$ can be joined by two lines on X defined over k.

Keeping track of the dependance on q allows us to deduce further results regarding the geometry of the moduli space $\mathrm{Mor}_{d,a,b}(\mathbb{P}_k^1, X)$, in the spirit of those obtained by Browning–Vishe [16]. We can regard f in (3.1) under the conditions given by (3.3) as a point in $\mathbb{P}_k^{n(d-1)-1}$. Then the space $\mathrm{Mor}_{d,a,b}(\mathbb{P}_k^1, X)$ is an open subvariety of $\mathbb{P}_k^{n(d-1)-1}$ cut out by $3d-1$ equations and so has expected naive dimension $\mu = (n-3)d - n$.

Corollary 3.1.3. *Fix $k = \mathbb{F}_q$ of $\mathrm{char}(k) > 3$. Fix a smooth cubic hypersurface $X \subset \mathbb{P}_k^{n-1}$, where $n \geqslant 10$. Pick any points $a, b \in X(k)$, not both on the Hessian hypersurface. Then for $d \geqslant \frac{19n-11}{n-9}$ we have*

$$\lim_{q\to\infty} q^{-\hat{\mu}} N_{a,b}(d) \leqslant 1,$$

where $\hat{\mu} = \mu + 1$.

A result similar to [16, Theorem 2.1] concerning $\mathrm{Mor}_{d,a,b}(\mathbb{P}_k^1, X)$ follows from Corollary 3.1.3.

Corollary 3.1.4. *Fix $k = \mathbb{F}_q$ of $\mathrm{char}(k) > 3$. Fix a smooth cubic hypersurface $X \subset \mathbb{P}_k^{n-1}$, where $n \geqslant 10$. Pick any points $a, b \in X(k)$, not both on the Hessian hypersurface. Then for $d \geqslant \frac{19n-11}{n-9}$ we have*

$$\lim_{q\to\infty} q^{-\mu} \# \mathrm{Mor}_{d,a,b}(\mathbb{P}_k^1, X)(k) \leqslant 1.$$

Now, by [47, Theorem II.1.2], all irreducible components of $\mathrm{Mor}_{d,a,b}(\mathbb{P}_k^1, X)$ have dimension at least μ. Then, comparing this with the Lang–Weil estimate [54], we obtain that the space $\mathrm{Mor}_{d,a,b}(\mathbb{P}_k^1, X)$ is irreducible and of expected dimension μ.

Following the same "spreading out" argument (see [32, §10.4.11] and [76]) as in [16, §2], the problem over $\mathbb{C}$ can be related to the problem over $\mathbb{F}_q$. More precisely, suppose $Y \in \mathbb{P}_{\mathbb{C}}^{n-1}$ is a smooth cubic hypersurface defined over $\mathbb{C}$, $n \geqslant 10$, and $a, b \in X(\mathbb{C})$, not both on the Hessian hypersurface. By (3.2) and (3.3), we can explicitly write equations defining $\mathrm{Mor}_{d,a,b}(\mathbb{P}^1_{\mathbb{C}}, Y)$. Thus, both Y and $\mathrm{Mor}_{d,a,b}(\mathbb{P}^1_{\mathbb{C}}, Y)$ may be viewed as schemes over the finitely generated $\mathbb{Z}$-algebra Λ obtained by adjoining to $\mathbb{Z}$ all coefficients of the defining equations of Y. The structure morphisms are $Y \to \operatorname{Spec}\Lambda$ and $\mathrm{Mor}_{d,a,b}(\mathbb{P}^1_{\mathbb{C}}, Y) \to \operatorname{Spec}\Lambda$. If U is a non-empty open set of $\operatorname{Spec}\Lambda$ and $\mathfrak{m} \in U$ is a closed point, then let $\mathrm{Mor}_{d,a,b}(\mathbb{P}^1_{\mathbb{C}}, Y)_{\mathfrak{m}}$ denote the fibre above $\mathfrak{m}$ obtained via the base change $\operatorname{Spec}\Lambda/\mathfrak{m} \to \operatorname{Spec}\Lambda$. We remark that if $\mathrm{Mor}_{d,a,b}(\mathbb{P}^1_{\mathbb{C}}, Y)_{\mathfrak{m}}$ is irreducible, then $\mathrm{Mor}_{d,a,b}(\mathbb{P}^1_{\mathbb{C}}, Y)$ is as well. Following the same idea as in [16], we use the Chevalley upper semicontinuity theorem [32, Theorem 13.1.3] to argue that there exists a non-empty open subset V of $\operatorname{Spec}\Lambda$ such that $\dim \mathrm{Mor}_{d,a,b}(\mathbb{P}^1_{\mathbb{C}}, Y) \leqslant \dim \mathrm{Mor}_{d,a,b}(\mathbb{P}^1_{\mathbb{C}}, Y)_{\mathfrak{m}}$, for any closed point $\mathfrak{m}$ in V. We may choose $\mathfrak{m}$ to be a maximal ideal in V, so that $\Lambda/\mathfrak{m}$ is a finite field, and then we may enlarge Λ, if necessary, so that it contains $\frac{1}{3}$. Thus, we have $\mathrm{char}(\Lambda/\mathfrak{m}) = p > 3$. It follows that $Y_{\mathfrak{m}}$ and $\mathrm{Mor}_{d,a,b}(\mathbb{P}^1_{\mathbb{C}}, Y)_{\mathfrak{m}}$ are quasi-projective varieties defined over $\overline{\mathbb{F}_p}$ obtained by reducing the coefficients of the defining equations of Y and $\mathrm{Mor}_{d,a,b}(\mathbb{P}^1_{\mathbb{C}}, Y)$, respectively, modulo $\mathfrak{m}$. We may assume $Y_{\mathfrak{m}}$ is smooth by further enlarging Λ, if needed. Hence, there exists a finite field such that both $Y_{\mathfrak{m}}$ and $\mathrm{Mor}_{d,a,b}(\mathbb{P}^1_{\mathbb{C}}, Y)_{\mathfrak{m}}$ are defined over it. Thus, Corollary 3.1.4 and its stated consequences lead to the following corollary.

Corollary 3.1.5. *Fix a smooth cubic hypersurface $X \subset \mathbb{P}^{n-1}$ defined over $\mathbb{C}$, where $n \geqslant 10$. Pick any points two points in $X(\mathbb{C})$, not both on the Hessian hypersurface. Then for each $d \geqslant \frac{19n-11}{n-9}$, the space $\mathcal{M}_{0,2}(\mathbb{P}^1_{\mathbb{C}}, X, d)$ is irreducible and of expected dimension $\bar{\mu} = \mu - 3$.*

In the case of stable maps, Harris–Roth–Starr [38] prove that for a general hypersurface $X \subset \mathbb{P}^{n-1}_{\mathbb{C}}$ of degree at most $n-2$, the Kontsevich moduli space $\overline{\mathcal{M}}_{0,m}(\mathbb{P}^1_{\mathbb{C}}, X, d)$ is a generically smooth, irreducible local complete intersection stack of the expected dimension.

3.2 THE CIRCLE METHOD OVER FUNCTION FIELDS

The notation used in this chapter has been established in Section 2.6; however, we give a brief recollection. Let $k = \mathbb{F}_q$ be a finite field of characteristic > 3, $K = k(t)$, and $\mathcal{O} = k[t]$. Finite primes ϖ in $\mathcal{O}$ are monic irreducible polynomials and we let $s = t^{-1}$ be the prime at infinity. These have associated absolute values which extend to give absolute values $|\cdot|_\varpi$ and $|\cdot| = |\cdot|_\infty$ on K. We let K_ϖ and K_∞ be the completions. Set $\mathbb{T} = \{\sum_{i\leqslant -1} a_i t^i | a_i \in \mathbb{F}_q\}$ and let $\widehat{R} = q^R$ for any real number R. There is a (Haar) measure on K_∞, and so on $\mathbb{T}$, which we normalise such that $\int_{\mathbb{T}} \mathrm{d}\alpha = 1$ and is extended to K_∞ in such a way that

$$\int_{\{\alpha\in K_\infty : |\alpha| < \widehat{N}\}} \mathrm{d}\alpha = \widehat{N},$$

for any positive integer N. Moreover, this can be extended to $\mathbb{T}^n$ and K_∞^n for any $n \in \mathbb{Z}_{>0}$. Denote by $\psi : K_\infty \to \mathbb{C}^*$ the non-trivial additive character on K_∞, given by

$$\sum_{i\leqslant N} a_i t^i \mapsto \exp\left(\frac{2\pi i \operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(a_{-1})}{p}\right),$$

where q is a power of p. These satisfy the orthogonality property in Lemma 2.1.2 initially proved by Kubota [52, Lemma 7].

Throughout this chapter, $S \ll T$ denotes an estimate of the form $S \leqslant CT$, where C is some constant that does not depend on q. Similarly, the implied constants in the notation $S = O(T)$ are independent of q. We start by proving the following technical result that will be necessary throughout this chapter.

Lemma 3.2.1. *Let $Y \in \mathbb{N}$. Then*

$$\sum_{\substack{m\in\mathcal{O}\\ |m|\leqslant \widehat{Y}\\ m \text{ monic}}} \frac{1}{|m|} = Y + 1.$$

Proof. We have

$$\sum_{\substack{m\in\mathcal{O}\\ |m|\leqslant \widehat{Y}\\ m \text{ monic}}} \frac{1}{|m|} = \sum_{n=0}^{Y} \frac{1}{q^n} \#\{m \in \mathcal{O} : |m| = q^n, m \text{ monic}\} = Y + 1,$$

as claimed. □

Recall that $F \in k[x_1, \ldots, x_n]$ denotes a non-singular homogeneous polynomial of degree 3. Moreover, let $X \subset \mathbb{P}_k^{n-1}$ be the smooth cubic hypersurface defined by $F = 0$, and let $\mathbf{a}, \mathbf{b} \in \mathbb{F}_q^n \backslash \{\mathbf{0}\}$ such that $F(\mathbf{a}) = F(\mathbf{b}) = 0$, $H(\mathbf{b}) \neq 0$. We want $x_i \in \mathbb{F}_q[t]$ such that $F(\mathbf{x}) = 0$ and

$$\mathbf{x}(0) = \mathbf{a}, \tag{3.4}$$

$$\mathbf{x}(\infty) = \mathbf{b}. \tag{3.5}$$

Now, t is a prime in $\mathbb{F}_q[t]$, and $s = t^{-1}$ is the prime at infinity. Moreover,

$$x_i = \sum_{0 \leqslant j \leqslant d} x_{ij} t^j = t^d \left(\sum_{0 \leqslant j \leqslant d} x_{ij} t^{j-d} \right) = t^d \left(\sum_{0 \leqslant j \leqslant d} x_{ij} s^{d-j} \right) = t^d y_i,$$

say, for $x_{ij} \in \mathbb{F}_q$. Then $y_i = t^{-d} x_i$ and (3.4) is equivalent to $\mathbf{x} \equiv \mathbf{a} \bmod t$, while (3.5) is equivalent to $\mathbf{y} \equiv \mathbf{b} \bmod s$. We remark that in order to generalise Theorem 3.1.1 to handle rational curves passing through more that two points in $X(k)$ we could let $\mathbf{a}_0, \ldots, \mathbf{a}_{q-1} \mathbb{F}_q^n \backslash \{\mathbf{0}\}$ such that $F(\mathbf{a}_l) = 0$ for all $0 \leqslant l \leqslant q-1$, but not all on the Hessian hypersurface. Then we look for $x_i \in \mathbb{F}_q[t]$ such that $F(\mathbf{x}) = 0$ and $\mathbf{x}(l) = \mathbf{a}_l$, i.e. $\mathbf{x} \equiv \mathbf{a}_l \bmod t + l$, for all $0 \leqslant l \leqslant q-1$.

Define a weight function $\omega : K_\infty^n \to \mathbb{R}_{\geqslant 0}$ such that

$$\omega(\mathbf{x}) = \begin{cases} 1, & \text{if } |t\mathbf{x} - \mathbf{b}| < 1, \\ 0, & \text{otherwise.} \end{cases}$$

This is the weight function $w(t^L(\mathbf{x} - \mathbf{x}_0))$ defined in [15, (7.2)], with $\mathbf{x}_0 = t^{-1}\mathbf{b}$ and $L = 1$. We now check that [15, (7.1)] and [15, (7.3)] hold. Recall that $H(\mathbf{x}) = \det \mathbf{H}(\mathbf{x})$ and note that $F(t^{-1}\mathbf{b}) = t^{-3}F(\mathbf{b}) = 0$, $H(t^{-1}\mathbf{b}) = H(t^{-1}\mathbf{b}) = t^{-n}H(\mathbf{b}) \neq 0$, and $|t^{-1}\mathbf{b}| = 1/q < 1$. Furthermore, $|\mathbf{x}| < 1$ for any $\mathbf{x} \in K_\infty^n$ such that $\omega(\mathbf{x}) \neq 0$. Moreover, any $\mathbf{x} \in K_\infty^n$ such that $\omega(\mathbf{x}) \neq 0$ can be written in the form $\mathbf{x} = t^{-1}(\mathbf{b} + \mathbf{z})$, where $\mathbf{z} \in \mathbb{T}^n$. Then, for any $\mathbf{x} \in K_\infty^n$ such that $\omega(\mathbf{x}) \neq 0$,

$$|\det \mathbf{H}(\mathbf{x})| = |H(t^{-1}(\mathbf{b} + \mathbf{z}))| = q^{-n}|H(\mathbf{b}) + \mathbf{z}.\nabla H(\mathbf{b}) + \ldots| = q^{-n},$$

since $H(\mathbf{b}) \in \mathbb{F}_q^*$ and $\mathbf{z} \in \mathbb{T}^n$. But $|H(t^{-1}\mathbf{b})| = |t^{-n}H(\mathbf{b})| = q^{-n}$, and thus $|H(\mathbf{x})| = |H(t^{-1}\mathbf{b})|$, for any $\mathbf{x} \in K_\infty^n$ such that $\omega(\mathbf{x}) \neq 0$. This confirms that [15, (7.1) and (7.3)] hold.

We have

$$N_{a,b}(d) = \sum_{\substack{\mathbf{x} \in \mathcal{O}^n \\ F(\mathbf{x}) = 0 \\ \mathbf{x} \equiv \mathbf{a} \bmod t}} \omega\left(\frac{\mathbf{x}}{t^{d+1}}\right),$$

where a and b are the corresponding points of $\mathbf{a}$ and $\mathbf{b}$ in $X(k)$. We remark that any $\mathbf{x}$ in the sum has $|\mathbf{x}| = q^d$. To simplify notation, we write $N_{a,b}(d) = N(d)$ and $P = t^{d+1}$. Then, $\omega(\mathbf{x}/P) \neq 0$ implies that $t^{-d}\mathbf{x} \equiv \mathbf{b} \bmod s$. Define

$$S(\alpha) = \sum_{\substack{\mathbf{x} \in \mathcal{O}^n \\ \mathbf{x} \equiv \mathbf{a} \bmod t}} \psi(\alpha F(\mathbf{x}))\omega\left(\frac{\mathbf{x}}{P}\right).$$

Then, by Lemma 2.1.3, we have

$$N(d) = \int_{\alpha \in \mathbb{T}} S(\alpha)\, \mathrm{d}\alpha.$$

By [15, Lemma 4.1], $\mathbb{T}$ can be partitioned into a union of intervals centred at rationals and since K is non-archimedean, the intervals do not overlap. Thus, for any $Q \geqslant 1$, we have

$$N(d) = \sum_{\substack{r \in \mathcal{O} \\ |r| \leqslant \widehat{Q} \\ r \text{ monic}}} \sideset{}{^*}\sum_{|a|<|r|} \int_{|\theta| < \frac{1}{|r|\widehat{Q}}} S\left(\frac{a}{r} + \theta\right) \mathrm{d}\theta, \tag{3.6}$$

where $\sum^*$ denotes a restriction to $(a, r) = 1$. We shall take $Q = \frac{3(d+1)}{2}$ in our work. We now note that $S(a/r + \theta)$ is the same as the exponential sum $S(a/r + \theta)$ appearing in [15, pg. 690], where $\mathbf{b}$ is $\mathbf{a}$ and $M = t$. Define $r_M = rM/(r, M)$, for any M,

$$S_{r,M,\mathbf{a}}(\mathbf{c}) = \sideset{}{^*}\sum_{|a|<|r|} \sum_{\substack{\mathbf{y} \in \mathcal{O}^n \\ |\mathbf{y}| < |r_M| \\ \mathbf{y} \equiv \mathbf{a} \bmod M}} \psi\left(\frac{aF(\mathbf{y})}{r}\right) \psi\left(\frac{-\mathbf{c}.\mathbf{y}}{r_M}\right), \tag{3.7}$$

$$I_r(\theta; \mathbf{c}) = \int_{K_\infty^n} \omega(\mathbf{u})\, \psi\left(\theta P^3 F(\mathbf{u}) + \frac{P\mathbf{c}.\mathbf{u}}{r}\right) \mathrm{d}\mathbf{u}. \tag{3.8}$$

Lemma 3.2.2. *Let $P = t^{d+1}$. We have*

$$N(d) = |P|^n \sum_{\substack{r \in \mathcal{O} \\ |r| \leqslant \widehat{Q} \\ r \text{ monic}}} |\tilde{r}|^{-n} \int_{|\theta| < \frac{1}{|r|\widehat{Q}}} \sum_{\substack{\mathbf{c} \in \mathcal{O}^n \\ |\mathbf{c}| \leqslant \widehat{C}}} S_{r,t,\mathbf{a}}(\mathbf{c})\, I_{\tilde{r}}(\theta; \mathbf{c})\, \mathrm{d}\theta, \tag{3.9}$$

where

$$\tilde{r} = \frac{rt}{(r,t)} = \begin{cases} rt, & \text{if } t \nmid r, \\ r, & \text{otherwise}, \end{cases}$$

and $\widehat{C} = q|\tilde{r}||P|^{-1} \max\left\{1, |\theta||P|^3\right\}$.

Proof. Applying [15, (7.7)] with $\mathbf{x}_0 = t^{-1}\mathbf{b}$ and $L = 1$, we have

$$I_{\tilde{r}}(\theta; \mathbf{c}) = \frac{1}{q^n} \psi\left(\frac{P\mathbf{c}.\mathbf{b}}{\tilde{r}t}\right) J_G\left(\theta P^3; \frac{Pt^{-1}\mathbf{c}}{\tilde{r}}\right),$$

where $G(\mathbf{v}) = F\left(t^{-1}\mathbf{b} + t^{-1}\mathbf{v}\right)$ and

$$J_G\left(\theta P^3; \frac{Pt^{-1}\mathbf{c}}{\tilde{r}}\right) = \int_{\mathbb{T}^n} \psi(\theta P^3 G(\mathbf{x}) + \frac{Pt^{-1}\mathbf{c}.\mathbf{x}}{\tilde{r}})\, \mathrm{d}\mathbf{x},$$

using the notation in [15, (2.4)]. According to [15, Lemma 2.6] we have

$$J_G\left(\theta P^3; \frac{Pt^{-1}\mathbf{c}}{\tilde{r}}\right) = 0$$

if $|\mathbf{c}| > q|\tilde{r}||P|^{-1} \max\left\{1, |\theta||P|^3\right\}$. Now apply [15, Lemma 4.4]. $\square$

We note that $C \in \mathbb{Z}$. Our strategy is now to go through the remaining arguments in [15, Sections 4 – 9] for our particular exponential sums and integrals, paying special attention to the uniformity in the q-aspect. Furthermore, we keep the same notation as in [15, Definition 4.6] for the factorisation of any $r \in \mathcal{O}$. Thus, for any $j \in \mathbb{Z}_{>0}$ we have $r = r_{j+1}\prod_{i=1}^{j} b_i = r_{j+1}\prod_{i=1}^{j} k_i^i$, with $(j+1)$-full r_{j+1}, where for any $i \in \mathbb{Z}_{>0}$ we have

$$b_i = \prod_{\varpi^i \| r} \varpi^i, \quad k_i = \prod_{\varpi^i \| r} \varpi, \quad u_i = \prod_{\substack{\varpi^e \| r \\ e \geqslant i}} \varpi^e.$$

3.2.1 *Exponential sum*

We continue to assume that $\mathrm{char}(\mathbb{F}_q) > 3$. Moreover, we note that $S_{r,M,\mathbf{a}}(\mathbf{c})$ satisfies the multiplicativity property recorded in [15, Lemma 4.5], which we restate below.

Lemma 3.2.3. *Let $r = r_1 r_2$ for coprime $r_1, r_2 \in \mathcal{O}$. Let $M = M_1 M_2 M_3$, where $M_1, M_2, M_3 \in \mathcal{O}$ such that $M_1 \mid r_1^{\infty}$. Then there exist non-zero $M_1 \mid r_1^{\infty}$, $M_2 \mid r_2^{\infty}$ and $(M_3, r) = 1$. Then there exist $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in k^n$, depending on $\mathbf{a}$, M and the residues of r_1, r_2 modulo M, such that*

$$S_{r,M,\mathbf{a}}(\mathbf{c}) = S_{r_1,M_1,\mathbf{a}_1}(\mathbf{c}) S_{r_2,M_2,\mathbf{a}_2}(\mathbf{c}) \psi\left(\frac{-\mathbf{c}.\mathbf{a}_3}{M_3}\right).$$

We record below the special case of Lemma 3.2.3 when $M \mid t$.

Lemma 3.2.4. *Let $r = r_1 r_2$ for coprime $r_1, r_2 \in \mathcal{O}$ and $t \nmid r_1$. Then there exist non-zero $\mathbf{a}', \mathbf{a}'' \in k^n$, depending on $\mathbf{a}$ and the residues of r_1, r_2 modulo t, such that*

$$S_{r,M,\mathbf{a}}(\mathbf{c}) = \begin{cases} S_{r_1,1,\mathbf{0}}(\mathbf{c}) S_{r_2,1,\mathbf{0}}(\mathbf{c}), & \text{if } M = 1, \\ S_{r_1,1,\mathbf{0}}(\mathbf{c}) S_{r_2,t,\mathbf{a}'}(\mathbf{c}), & \text{if } M = t \text{ and } t \mid r, \\ S_{r_1,1,\mathbf{0}}(\mathbf{c}) S_{r_2,1,\mathbf{0}}(\mathbf{c}) \psi\left(\frac{-\mathbf{c}.\mathbf{a}''}{t}\right), & \text{if } M = t \text{ and } t \nmid r. \end{cases}$$

Furthermore, the estimates in [15, Lemma 5.1], [15, (5.2)] and [15, (5.3)] all hold and are independent of q. Next we record the following result, which holds for any $M \in \mathcal{O}$, but we will only need for $M \mid t$.

Lemma 3.2.5. *Let $\mathbf{r} \in K_{\infty}^n$, $C \in \mathbb{N}$, $M \in \mathcal{O}$ and $\varepsilon > 0$. Then there exists a constant $c_{n,\varepsilon} > 0$, depending only on n and ε, such that*

$$\sum_{\substack{\mathbf{c} \in \mathcal{O}^n \\ |\mathbf{c}-\mathbf{r}| < \widehat{C}}} |S_{r_3,M,\mathbf{a}}(\mathbf{c})| \leqslant c_{n,\varepsilon} |M|^n |r_3|^{n/2+1+\varepsilon} \left(|r_3|^{n/3} + \widehat{C}^n\right).$$

Proof. This follows directly from [15, Lemma 6.4] on noting that $H_F = |\Delta_F| = 1$ in our situation. □

Let $F^* \in k[x_1, \ldots, x_n]$ be the dual form of F. Its zero locus parametrises the set of hyperplanes whose intersection with the cubic hypersurface $F = 0$ produces a singular variety. Moreover, F^* is absolutely irreducible and has degree $3 \cdot 2^{n-2}$. We shall need the following variation of [15, Lemma 6.4] in which the sum is restricted to zeros of F^*.

Lemma 3.2.6. *Let $C \in \mathbb{N}$, $M \in \{1, t\}$ and $\varepsilon > 0$. Then there exists a constant $c_{n,\varepsilon} > 0$, depending only on n and ε, such that*

$$\sum_{\substack{\mathbf{c} \in \mathcal{O}^n \\ |\mathbf{c}| < \widehat{C} \\ F^*(\mathbf{c}) = 0}} |S_{r_3, M, \mathbf{a}}(\mathbf{c})| \leqslant c_{n,\varepsilon} |M|^{\frac{2n-5}{3}+\varepsilon} |r_3|^{\frac{5n+7}{6}+\varepsilon} \left(1 + \widehat{C}\right)^{n - \frac{3}{2} + \varepsilon}.$$

Proof. This proof uses the same methods as in Section 7 of [40]. We remark that if $M = 1$ when $t \nmid r_3$, and $M = t$ when $t \mid r_3$. Thus, $r_{3_M} = r_3 M/(r_3, M) = r_3$. Hence, by (3.7), we have

$$S_{r_3, M, \mathbf{a}}(\mathbf{c}) = \sideset{}{^*}\sum_{|a| < |r_3|} \sum_{\substack{\mathbf{y} \in \mathcal{O}^n \\ |\mathbf{y}| < |r_3| \\ \mathbf{y} \equiv \mathbf{a} \bmod M}} \psi\left(\frac{aF(\mathbf{y}) - \mathbf{c}.\mathbf{y}}{r_3}\right).$$

Setting $\mathbf{y} = \mathbf{a} + M\mathbf{z}$, we have

$$S_{r_3, M, \mathbf{a}}(\mathbf{c}) = \psi\left(\frac{-\mathbf{c}.\mathbf{a}}{r_3}\right) \sideset{}{^*}\sum_{|a| < |r_3|} S_a(\mathbf{c}),$$

where

$$S_a(\mathbf{c}) = \sum_{\substack{\mathbf{z} \in \mathcal{O}^n \\ |\mathbf{z}| < |s|}} \psi\left(\frac{aM^{-1}F(\mathbf{a} + M\mathbf{z}) - \mathbf{c}.\mathbf{z}}{l}\right)$$

and $l = r_3/M$. Denote

$$g(\mathbf{z}) = M^{-1}F(\mathbf{a} + M\mathbf{z}), \tag{3.10}$$

write $l = c^2 d$, where d is square-free, $d \mid c$, and put $\mathbf{z} = \mathbf{z}_1 + cd\mathbf{z}_2$, with $|\mathbf{z}_1| < |cd|$. Then,

$$S_a(\mathbf{c}) = |c|^n \sum_{\substack{\mathbf{z}_1 \in \mathcal{O}^n \\ |\mathbf{z}_1| < |cd| \\ a\nabla g(\mathbf{z}_1) \equiv \mathbf{c} \bmod c}} \psi\left(\frac{ag(\mathbf{z}_1) - \mathbf{c}.\mathbf{z}_1}{c^2 d}\right).$$

Write $a = a_1 + Mca_2$ with $|a_1| < |Mc|$. Then $(a, r_3) = 1$ if and only if $(a_1, Mc) = 1$ and thus,

$$\sideset{}{^*}\sum_{|a| < |r_3|} S_a(\mathbf{c}) = |c|^n |cd| \sideset{}{^*}\sum_{|a_1| < |Mc|} \sum_{\substack{\mathbf{z}_1 \in \mathcal{O}^n \\ |\mathbf{z}_1| < |cd| \\ a_1 \nabla g(\mathbf{z}_1) \equiv \mathbf{c} \bmod c \\ g(\mathbf{z}_1) \equiv 0 \bmod cd}} \psi\left(\frac{a_1 g(\mathbf{z}_1) - \mathbf{c}.\mathbf{z}_1}{c^2 d}\right).$$

Writing $\mathbf{z}_1 = \mathbf{h} + c\mathbf{j}$ with $|\mathbf{h}| < |c|$, we have $g(\mathbf{z}_1) \equiv g(\mathbf{h}) + c\mathbf{j}\nabla g(\mathbf{h}) \bmod cd$ and $a_1\nabla g(\mathbf{z}_1) \equiv a_1\nabla g(\mathbf{h}) \bmod c$, since $cd \mid c^2$. Now, $g(\mathbf{z}_1) \equiv 0 \bmod cd$ is equivalent to $g(\mathbf{h}) + c\mathbf{j}\nabla g(\mathbf{h}) \equiv 0 \bmod cd$. Thus, $g(\mathbf{h}) \equiv 0 \bmod c$ and we can write $g(\mathbf{h}) = mc$. Thus, $m + \mathbf{j}\nabla g(\mathbf{h}) \equiv 0 \bmod d$. Moreover, if $a_1\nabla g(\mathbf{h}) = \mathbf{c} + c\mathbf{k}$, then $a_1 g(\mathbf{z}_1) - \mathbf{c}.\mathbf{z}_1 \equiv a_1 g(\mathbf{h}) - \mathbf{c}.\mathbf{h} + c^2(\mathbf{k}.\mathbf{j} + a_1\mathbf{h}\nabla g(\mathbf{j})) \bmod c^2 d$, and thus, the sum over $\mathbf{z}_1$ becomes

$$\sum_{\substack{\mathbf{h}\in\mathcal{O}^n\\|\mathbf{h}|<|c|\\a_1\nabla g(\mathbf{h})=\mathbf{c}+c\mathbf{k}\\g(\mathbf{h})=mc}} \psi\left(\frac{a_1 g(\mathbf{h}) - \mathbf{c}.\mathbf{h}}{c^2 d}\right) \sum_{\substack{\mathbf{j}\in\mathcal{O}^n\\|\mathbf{j}|<|d|\\m+\mathbf{j}\nabla g(\mathbf{h})\equiv 0 \bmod d}} \psi\left(\frac{\mathbf{k}.\mathbf{j} + a_1\mathbf{h}\nabla g(\mathbf{j})}{d}\right).$$

Denote the sum over $\mathbf{j}$ by $S_{\mathbf{k},\mathbf{h}}$ and estimate it by writing

$$|S_{\mathbf{k},\mathbf{h}}|^2 = \sum_{\substack{\mathbf{j}_1,\mathbf{j}_2\in\mathcal{O}^n\\|\mathbf{j}_1|,|\mathbf{j}_2|<|d|\\\mathbf{j}_1\nabla g(\mathbf{h})\equiv\mathbf{j}_2\nabla g(\mathbf{h}) \bmod d}} \psi\left(\frac{\mathbf{k}.(\mathbf{j}_1 - \mathbf{j}_2) + a_1\mathbf{h}(\nabla g(\mathbf{j}_1) - \nabla g(\mathbf{j}_2))}{d}\right).$$

Writing $\mathbf{j}_1 = \mathbf{j}_2 + \mathbf{j}_3$ and recalling (3.10), we note that

$$\mathbf{h}\left(\nabla g(\mathbf{j}_1) - \nabla g(\mathbf{j}_2)\right) = \frac{1}{2}\mathbf{j}_3^T\nabla^2 g(\mathbf{h})\mathbf{j}_3 + \mathbf{j}_2^T\nabla^2 g(\mathbf{h})\mathbf{j}_3$$

and therefore,

$$|S_{\mathbf{k},\mathbf{h}}|^2 \leqslant \sum_{\substack{\mathbf{j}_3\in\mathcal{O}^n\\|\mathbf{j}_3|<|d|\\\mathbf{j}_3\nabla g(\mathbf{h})\equiv 0 \bmod d}} \left|\sum_{\substack{\mathbf{j}_2\in\mathcal{O}^n\\|\mathbf{j}_2|<|d|}} \psi\left(\frac{\mathbf{j}_2.(a_1\nabla^2 g(\mathbf{h})\mathbf{j}_3)}{d}\right)\right| \leqslant |d|^n M_d(\mathbf{h}),$$

where $M_d(\mathbf{h}) = \#\left\{\mathbf{j}_3 \in \mathcal{O}^n : |\mathbf{j}_3| < |d|, \nabla^2 g(\mathbf{h})\mathbf{j}_3 \equiv 0 \bmod d\right\}$. Thus,

$$\sum_{\substack{\mathbf{c}\in\mathcal{O}^n\\|\mathbf{c}|<\widehat{C}\\F^*(\mathbf{c})=0}} |S_{r_3,M,\mathbf{a}}(\mathbf{c})| \leqslant |c|^{n+2}|d|^{n/2+1}|M| \sum_{\substack{\mathbf{c}\in\mathcal{O}^n\\|\mathbf{c}|<\widehat{C}\\F^*(\mathbf{c})=0}} \sum_{\substack{\mathbf{h}\in\mathcal{O}^n\\|\mathbf{h}|<|c|\\g(\mathbf{h})\equiv 0 \bmod c}} M_d(\mathbf{h})^{1/2}.$$

Now we make the change of variables $\mathbf{x} = M\mathbf{h} + \mathbf{a}$ and note that there exist elements $c' = cM$ and $d' = \frac{d}{(d,M)}$ with $d' \mid c'$, such that

$$\sum_{\substack{\mathbf{h}\in\mathcal{O}^n\\|\mathbf{h}|<|c|\\g(\mathbf{h})\equiv 0 \bmod c}} M_d(\mathbf{h})^{1/2} \leqslant |(d,M)|^{n/2} \sum_{\substack{\mathbf{x}\in\mathcal{O}^n\\|\mathbf{x}|<|c'|\\F(\mathbf{x})\equiv 0 \bmod c'}} N_{d'}(\mathbf{x})^{1/2},$$

where $N_{d'}(\mathbf{x}) = \#\left\{\mathbf{y} \in \mathcal{O}^n : |\mathbf{y}| < |d'|, \mathbf{H}(\mathbf{x})\mathbf{y} \equiv 0 \bmod d'\right\}$. Thus,

$$\sum_{\substack{\mathbf{c}\in\mathcal{O}^n\\|\mathbf{c}|<\widehat{C}\\F^*(\mathbf{c})=0}} |S_{r_3,M,\mathbf{a}}(\mathbf{c})| \leqslant |c|^{n+2}|d|^{n/2+1}|M||(d,M)|^{n/2}\mathcal{N} \sum_{\substack{\mathbf{x}\in\mathcal{O}^n\\|\mathbf{x}|<|c'|\\F(\mathbf{x})\equiv 0 \bmod c'}} N_{d'}(\mathbf{x})^{1/2},$$

where

$$\mathcal{N} := \#\left\{\mathbf{c} \in \mathcal{O}^n : |\mathbf{c}| < \widehat{C}, F^*(\mathbf{c}) = 0\right\} \ll \left(1 + \widehat{C}\right)^{n-3/2+\varepsilon},$$

for any $\varepsilon > 0$, by [15, Lemma 2.10].

It remains to bound the inner sum. As is [15], let

$$S(c,d) = \sum_{\substack{\mathbf{x}\in\mathcal{O}^n \\ |\mathbf{x}|<|c| \\ F(\mathbf{x})\equiv 0 \bmod c}} N_d(\mathbf{x})^{1/2},$$

for given c, d in $\mathcal{O}$, where $d \mid c$ and d is square-free. This sum satisfies a multiplicativity property, i.e. for any c_i, d_i in $\mathcal{O}$ such that $(c_1d_1, c_2d_2) = 1$ and $d_i \mid c_i$ we have $S(c_1c_2, d_1d_2) = S(c_1, d_1)S(c_2, d_2)$. Thus, we only need to look at the cases when $c = \varpi^e$ and $d = 1$, and $c = \varpi^e$ and $d = \varpi$, for any $e \in \mathbb{Z}_{>0}$ and any prime ϖ. Note that F is non-singular modulo any prime ϖ.

The arguments that follow are similar to [40, p. 244]. Define

$$S_0(\varpi^e) = \#\left\{\mathbf{x} \in \mathcal{O}^n : |\mathbf{x}| < |\varpi|^e, F(\mathbf{x}) \equiv 0 \bmod \varpi^e\right\},$$
$$S_1(\varpi^e) = \#\left\{\mathbf{x} \in \mathcal{O}^n : |\mathbf{x}| < |\varpi|^e, \varpi \nmid \mathbf{x}, F(\mathbf{x}) \equiv 0 \bmod \varpi^e\right\},$$

for $e \geqslant 1$. Then, as in [40, (7.4), (7.5)], we have

$$S_0(\varpi^e) = S_1(\varpi^e) + |\varpi|^{2n} S_0(\varpi^{e-3}), \text{ for } e \geqslant 4, \tag{3.11}$$

$$S_0(\varpi^e) = S_1(\varpi^e) + |\varpi|^{(e-1)n}, \text{for } 1 \leqslant e \leqslant 3, \tag{3.12}$$

$$S_1(\varpi^{e+1}) = |\varpi|^{n-1} S_1(\varpi^e), \text{for } e \geqslant 1. \tag{3.13}$$

Since, $N_1(\mathbf{x}) = 1$, we have $S(\varpi^e, 1) = S_0(\varpi^e)$. Moreover, $S_1(\varpi) \ll |\varpi|^{n-1}$, and thus,

$$S_1(\varpi^e) \ll |\varpi|^{e(n-1)}, \tag{3.14}$$

for $e \geqslant 1$. Thus, for $1 \leqslant e \leqslant 3$ and $n \geqslant 4$, we have $S_0(\varpi^e) \ll |\varpi|^{e(n-1)}$. Similarly, for $e \geqslant 4$ and $n \geqslant 4$, we can use an induction argument to get $S_0(\varpi^e) \ll |\varpi|^{e(n-1)}$. Thus, for $c = \varpi^e$ and $d = 1$, $S(c,d) \leqslant A_1^{\omega(c)}|c|^{n-1}$. Consider now the case when $c = \varpi^e$ and $d = \varpi$. After a change of variables, $S(\varpi^e, \varpi)$ is equal to

$$\sum_{\substack{\mathbf{z}\in\mathcal{O}^n \\ |\mathbf{z}|<|\varpi|}} N_\varpi(\mathbf{z})^{1/2} \#\left\{\mathbf{x} \in \mathcal{O}^n : |\mathbf{x}| < |\varpi|^e, \mathbf{x} \equiv \mathbf{z} \bmod \varpi, F(\mathbf{x}) \equiv 0 \bmod \varpi^e\right\}.$$

First analyse the contribution to $S(\varpi^e,\varpi)$ coming from $\mathbf{z}$ such that $\varpi\nmid\mathbf{z}$. Then, as in [40], by Cauchy's inequality, it follows that this contribution is

$$\ll |\varpi|^{(e-1)(n-1)} \sum_{\substack{\mathbf{z}\in\mathcal{O}^n \\ |\mathbf{z}|<|\varpi| \\ \varpi\nmid\mathbf{z} \\ F(\mathbf{z})\equiv 0 \bmod \varpi}} N_\varpi(\mathbf{z})^{1/2} \leqslant |\varpi|^{(e-1)(n-1)} S_{N_\varpi}(\mathbf{z})^{1/2} S_0(\varpi)^{1/2},$$

where

$$S_{N_\varpi}(\mathbf{z}) = \#\left\{\mathbf{z},\mathbf{y}\in\mathcal{O}^n : |\mathbf{z}|<|\varpi|, |\mathbf{y}|<|\varpi|, \varpi\nmid\mathbf{z}, \mathbf{H}(\mathbf{z}).\mathbf{y}\equiv\mathbf{0} \bmod \varpi\right\}.$$

Then, by (3.13), (3.14) and [40, Lemma 4], there exists some constant A such that the contribution to $S(\varpi^e,\varpi)$ coming from $\mathbf{z}$ such that $\varpi\nmid\mathbf{z}$ is

$$\leqslant A^{\omega(\varpi)}|\varpi|^{(e-1)(n-1)}|\varpi|^{\frac{n}{2}}|\varpi|^{\frac{n-1}{2}} = A|\varpi|^{e(n-1)+\frac{1}{2}}.$$

The remaining contribution to $S(\varpi^e,\varpi)$ comes from $\mathbf{z}=\mathbf{0}$. In this case, $N_\varpi(\mathbf{0}) = |\varpi|^n$, and thus this contribution is

$$|\varpi|^{\frac{n}{2}}\#\left\{\mathbf{y}\in\mathcal{O}^n : |\mathbf{y}|<|\varpi|^{e-1}, \varpi^3F(\mathbf{y})\equiv 0 \bmod \varpi^e\right\}. \tag{3.15}$$

Then, as in [40], if $1\leqslant e\leqslant 3$, (3.15) becomes

$$|\varpi|^{\frac{n}{2}}\#\left\{\mathbf{y}\in\mathcal{O}^n : |\mathbf{y}|<|\varpi|^{e-1}\right\} = |\varpi|^{n(e-1/2)}, \tag{3.16}$$

and if $4\leqslant e$, there exists a constant A such that (3.15) is equal to

$$|\varpi|^{\frac{5n}{2}}S_0(\varpi^{e-3}) \leqslant A^{\omega(\varpi)}|\varpi|^{\frac{5n}{2}+(e-3)(n-1)} = A|\varpi|^{e(n-1)-\frac{n}{2}+3}. \tag{3.17}$$

Note that if $n\geqslant 5$, then the contributions in (3.16) and (3.17) are both $\ll |\varpi|^{e(n-1)+\frac{1}{2}}$, and thus $S(c,d)\ll A^{\omega(c)}|c|^{n-1}|d|^{1/2}$.

Putting everything together, we have

$$\sum_{\substack{\mathbf{c}\in\mathcal{O}^n \\ |\mathbf{c}|<\widehat{C} \\ F^*(\mathbf{c})=0}} |S_{r_3,M,\mathbf{a}}(\mathbf{c})| \leqslant |c|^{n+2}|d|^{\frac{n}{2}+1}|M||(d,M)|^{\frac{n}{2}}A^{\omega(c')}|c'|^{n-1}|d'|^{\frac{1}{2}}\left(1+\widehat{C}\right)^{n-\frac{3}{2}+\varepsilon}.$$

Replacing c', d' by their definition and using $c^2d = r_3/M$, followed by an application of Lemma 2.2.5 concludes the proof. □

3.2.2 *Exponential integral*

The following result is similar to [15, Lemma 7.3]. It gives a good upper bound for $I_{\tilde{r}}(\theta;\mathbf{c})$, for $r,\theta,\mathbf{c}$ appearing in the expression for $N(d)$ in Lemma 3.2.2.

Lemma 3.2.7. *We have*

$$|I_{\tilde{r}}(\theta;\mathbf{c})| \ll \min\left\{q^{-n}, q^n|\theta P^3|^{-n/2}\right\},$$

where the implicit constant is independent of q.

Proof. As in [15, Lemma 7.3], we have $|I_{\tilde{r}}(\theta;\mathbf{c})| \leqslant \operatorname{meas}(\mathcal{R})$, where

$$\mathcal{R} = \left\{\mathbf{x}\in\mathbb{T}^n : |t\mathbf{x}-\mathbf{b}|<1, |\theta P^3\nabla F(\mathbf{x}) + Pt^{-1}\mathbf{c}/\tilde{r}| \leqslant \max\left\{1, |\theta P^3|^{1/2}\right\}\right\}.$$

If $|\theta P^3| \leqslant 1$, then we have the trivial bound $\operatorname{meas}(\mathcal{R}) \leqslant q^{-n}$. Otherwise, given $\mathbf{x}\in\mathcal{R}$, we can write it as $\mathbf{x} = \mathbf{b}t^{-1}+\mathbf{d}$, where $\mathbf{d}\in\mathbb{T}^n$, $|\mathbf{d}|\leqslant q^{-2}$. Then

$$|H(\mathbf{x})| = |t^{-n}H(\mathbf{b}) + \mathbf{d}.\nabla H(\mathbf{b}t^{-1}) + \ldots| = q^{-n}(1+O(q^{-1})).$$

Since the entries in the adjugate of $\mathbf{H}(\mathbf{x})$ have norms equal to $q^{-n+1}(1+O(q^{-1}))$, the inverse of $\mathbf{H}(\mathbf{x})$ has entries with absolute value $q+O(1)$. Thus, if $\mathbf{x}$ and $\mathbf{x}+\mathbf{x}'$ are in $\mathcal{R}$, we have $|\mathbf{x}'| \ll q|\theta P^3|^{-1/2}$ and thus, $\operatorname{meas}(\mathcal{R}) \ll q^n|\theta P^3|^{-n/2}$. □

Note that this result uses crucially the condition that one of the two fixed points in Theorem 3.1.1 does not lie on the Hessian of X.

3.3 THE MAIN TERM

In this section, we investigate the contribution to $N(d)$ in Lemma 3.2.2 coming from $\mathbf{c}=\mathbf{0}$. Preserving the notation in [15], let us denote this term by $M(d)$. We will always assume $n\geqslant 10$. Thus,

$$M(d) = |P|^n \sum_{\substack{r\in\mathcal{O} \\ |r|\leqslant\widehat{Q} \\ r \text{ monic}}} |\tilde{r}|^{-n} S_{r,t,\mathbf{a}}(\mathbf{0})\, K_r,$$

where

$$K_r = \int_{|\theta|<\frac{1}{|r|\widehat{Q}}} I_{\tilde{r}}(\theta;\mathbf{0})\,\mathrm{d}\theta.$$

Recall that $\nabla F(t^{-1}\mathbf{b}) \neq \mathbf{0}$ and, in particular, $q^{-2} = |\nabla F(t^{-1}\mathbf{b})|$. This corresponds to taking $\xi = -2$ in [15, Section 7.3]. The following result gives a similar bound to that in [15, Lemma 7.4].

Lemma 3.3.1. *For any $Y \in \mathbb{N}$ and any $\varepsilon > 0$ we have*

$$\sum_{\substack{r\in\mathcal{O} \\ |r|=\widehat{Y} \\ r \text{ monic}}} |\tilde{r}|^{-n} \left|S_{r,t,\mathbf{a}}\left(\mathbf{0}\right)\right| \ll q^{2n}\widehat{Y}^{-\frac{n}{6}+\frac{4}{3}+\varepsilon}(q^{-n}+\widehat{Y}^{\frac{3-n}{3}}),$$

where the implicit constant is independent of q.

Proof. Write $r = b_1b_2r_3$. Then, by the multiplicativity property in Lemma 3.25, we have

$$\left|S_{r,t,\mathbf{a}}\left(\mathbf{0}\right)\right| = \left|S_{b_1b_2,M,\mathbf{a}'}\left(\mathbf{0}\right)\right|\left|S_{r_3,M_3,\mathbf{a}''}\left(\mathbf{0}\right)\right|,$$

where $M, M_3 \in \{1, t\}$ and $\mathbf{a}', \mathbf{a}'' \in k^n$ depend on $\mathrm{ord}_t(r)$. By [15, Lemma 5.1],

$$\left|S_{b_1b_2,M,\mathbf{a}}\left(\mathbf{0}\right)\right| \ll |b_1b_2|^{\frac{n}{2}+1+\varepsilon},$$

where the implicit constant is independent of q. Moreover,

$$\left|S_{r_3,M_3,\mathbf{a}}\left(\mathbf{0}\right)\right| \leqslant \sum_{\substack{\mathbf{c}\in\mathcal{O}^n \\ |\mathbf{c}|<\widehat{C}}} \left|S_{r_3,M_3,\mathbf{a}}(\mathbf{c})\right|,$$

for any $C > 0$. Taking $C = 1$ and using [15, Lemma 6.4], we get

$$\left|S_{r_3,M_3,\mathbf{a}}\left(\mathbf{0}\right)\right| \leqslant \sum_{\substack{\mathbf{c}\in\mathcal{O}^n \\ |\mathbf{c}|<q}} \left|S_{r_3,M_3,\mathbf{a}}(\mathbf{c})\right| \ll |M_3|^n|r_3|^{n/2+1+\varepsilon}\left(|r_3|^{n/3}+q^n\right),$$

where the the implicit constant depends only on n and ε. On noting that for $|r| = \widehat{Y}$, we have $|\tilde{r}|^{-n} = q^{-n}\widehat{Y}^{-n}$ if $t \nmid r$ and $|\tilde{r}|^{-n} = \widehat{Y}^{-n}$, otherwise, we obtain

$$\sum_{\substack{r\in\mathcal{O} \\ |r|=\widehat{Y} \\ r \text{ monic}}} |\tilde{r}|^{-n} \left|S_{r,t,\mathbf{a}}\left(\mathbf{0}\right)\right| \ll |M_3|^n\widehat{Y}^{-\frac{n}{2}+1+\varepsilon} \sum_{\substack{r_3\in\mathcal{O} \\ |r_3|\leqslant\widehat{Y} \\ r_3 \text{ monic}}} \left(|r_3|^{n/3}+q^n\right)\frac{\widehat{Y}}{|r_3|}.$$

Then, since $\#\left\{r_3 \in \mathcal{O} : |r_3| \leqslant \widehat{Y}\right\} = O(\widehat{Y}^{1/3})$ and $M_3 \in \{1, t\}$, we can bound the above by

$$\ll q^n\widehat{Y}^{-\frac{n}{2}+\frac{7}{3}+\varepsilon}\left(\widehat{Y}^{\frac{n}{3}-1}+q^n\right),$$

which concludes the proof. $\square$

Put $C = \widehat{L-\xi} = q^3$. Then, if $C^{-1}\widehat{Q} \leqslant |r| \leqslant \widehat{Q}$, we have $|\theta| < |r|^{-1}\widehat{Q}^{-1} \leqslant q^3|P|^{-3}$, and thus, $|\theta P^3| \leqslant q^2$. Then, by Lemma 3.2.7 we have $K_r = O(q^{3-n}|P|^{-3})$ in this case. On noting that the exponents of $\widehat{Y}$ in the bound given by Lemma 3.3.1 are negative for $n > 8$, we obtain that

$$\sum_{\substack{r\in\mathcal{O} \\ q^{-3}\widehat{Q}\leqslant|r|\leqslant\widehat{Q} \\ r \text{ monic}}} |P|^n|\tilde{r}|^{-n}S_{r,t,\mathbf{a}}(\mathbf{0})\,K_r \ll |P|^{n-3}q^{3+n}q^{(Q-3)(-\frac{n}{6}+\frac{4}{3}+\varepsilon)}(q^{-n}+q^{(Q-3)\frac{3-n}{3}}).$$

Thus, recalling that $\widehat{Q}^2 = |P|^3$, the contribution to $M(d)$ coming from such r is $\ll q^{11n/12+2/3}|P|^{3n/4-1+\varepsilon'}\left(1+q^{2n-3}|P|^{3/2-n/2}\right)$. If $|r| < q^{-3}\widehat{Q}$, as in [15, Section 7.3], K_r is independent of r. Moreover, we only get a contribution from $|\theta| < q^3|P|^{-3}$. Thus, for $d \geqslant 3(n-1)/(n-3)$, we have

$$M(d) = |P|^{n-3}\mathfrak{S}(Q)\mathfrak{J} + O(q^{11n/12+2/3}|P|^{3n/4-1}),$$

where

$$\mathfrak{S}(Q) = \sum_{\substack{r\in\mathcal{O} \\ |r|\leqslant\widehat{Q} \\ r \text{ monic}}} |\tilde{r}|^{-n}S_{r,t,\mathbf{a}}(\mathbf{0})$$

and

$$\mathfrak{J} = \int_{|\varphi|<q^3}\int_{K_\infty^n} w(t\mathbf{u}-\mathbf{b})\,\psi(\varphi F(\mathbf{x}))\,\mathrm{d}\mathbf{x}\,\mathrm{d}\varphi$$

is the *singular integral*. By taking $\mathbf{x}_0 = t^{-1}\mathbf{b}$ and $L = 1$ in [15, Lemma 7.5], it follows that

$$\mathfrak{J} = \frac{1}{q^{n-3}}. \tag{3.18}$$

By Lemma 3.3.1, we can extend the summation over r in $\mathfrak{S}(Q)$ to infinity with an acceptable error since

$$\mathfrak{S} - \mathfrak{S}(Q) \ll q^{\frac{5n}{6}+\frac{4}{3}}|P|^{-\frac{n}{4}+2+\varepsilon}\left(1+q^{\frac{2n+3}{3}}|P|^{2(3-n)}\right).$$

and thus,

$$|P|^{n-3}\left(\mathfrak{S}-\mathfrak{S}(Q)\right)\mathfrak{J} = |P|^{\frac{3n}{4}-1}q^{-\frac{n}{6}+\frac{13}{3}}\left(1+q^{\frac{2n+3}{3}}|P|^{2(3-n)}\right).$$

Then

$$\mathfrak{S} = \sum_{\substack{r\in\mathcal{O} \\ r \text{ monic}}} |\tilde{r}|^{-n}S_{r,t,\mathbf{a}}(\mathbf{0})$$

is the absolutely convergent *singular series*.

Lemma 3.3.2. *We have*

$$\mathfrak{S} = q^{-n+1} + O(q^{-3n/2+3}).$$

Proof. First, recalling the definition of $\tilde{r}$, decompose $\mathfrak{S}$ into

$$\mathfrak{S} = q^{-n} \sum_{\substack{r \in \mathcal{O} \\ r \text{ monic} \\ (r,t)=1}} |r|^{-n} S_{r,t,\mathbf{a}}(\mathbf{0}) + \sum_{\substack{r \in \mathcal{O} \\ r \text{ monic} \\ t|r}} |r|^{-n} S_{r,t,\mathbf{a}}(\mathbf{0}). \tag{3.19}$$

Then note that by the multiplicativity property in Lemma 3.25, given $r = t^A \prod_{\varpi \neq t} \varpi^e \in \mathcal{O}$, where $A \in \mathbb{Z}_{\geqslant 0}$ and ϖ are primes in $\mathcal{O}$, we have

$$S_{r,t,\mathbf{a}}(\mathbf{0}) = S_{t^A,t,\mathbf{a}}(\mathbf{0}) \prod_{\substack{\varpi \text{ prime} \\ \varpi \neq t \\ \varpi^e || r}} S_{\varpi^e}(\mathbf{0}),$$

where $S_{\varpi^e}(\mathbf{0}) = S_{\varpi^e,1,\mathbf{0}}(\mathbf{0}) = S_{\varpi^e,1,\mathbf{a}}(\mathbf{0})$. Thus,

$$\mathfrak{S} = \left(q^{-n} + \sum_{A=1}^{\infty} q^{-An} S_{t^A,t,\mathbf{a}}(\mathbf{0}) \right) \prod_{\substack{\varpi \text{ prime} \\ \varpi \neq t}} \sum_{e=0}^{\infty} |\varpi|^{-en} S_{\varpi^e}(\mathbf{0}).$$

Now, by (3.7),

$$S_{t,t,\mathbf{a}}(\mathbf{0}) = \sum_{|a|<|t|}^{*} \sum_{\substack{\mathbf{y} \in \mathcal{O}^n \\ |\mathbf{y}|<|t| \\ \mathbf{y} \equiv \mathbf{a} \bmod t}} \psi\left(\frac{aF(\mathbf{y})}{t}\right) = \sum_{|a|<|t|}^{*} \psi\left(\frac{aF(\mathbf{a})}{t}\right) = \sum_{|a|<|t|}^{*} 1 = q-1,$$

since $F(\mathbf{a}) = 0$. Similarly, by (3.7), after making the change of variables $\mathbf{y} = \mathbf{a} + t\mathbf{z}$, we have

$$S_{t^2,t,\mathbf{a}}(\mathbf{0}) = \sum_{|a|<|t|^2}^{*} \sum_{\substack{\mathbf{z} \in \mathcal{O}^n \\ |\mathbf{z}|<|t|}} \psi\left(\frac{a\mathbf{z}.\nabla F(\mathbf{a})}{t}\right) = \sum_{\substack{|a|<|t|^2 \\ a\nabla F(\mathbf{a}) \equiv 0 \bmod t}}^{*} q^n = 0,$$

Moreover, for $K \geqslant 3$, we have

$$\begin{aligned} S_{t^K,t,\mathbf{a}}(\mathbf{0}) &= \sum_{\substack{\mathbf{y} \in \mathcal{O}^n \\ |\mathbf{y}|<|t|^K \\ \mathbf{y} \equiv \mathbf{a} \bmod t}} \left(\sum_{|a_1|<|t|^K} \psi\left(\frac{a_1 F(\mathbf{y})}{t^K}\right) - \sum_{|a_2|<|t|^{K-1}} \psi\left(\frac{a_2 F(\mathbf{y})}{t^{K-1}}\right) \right) \\ &= q^K S_{\mathbf{a}}(K) - q^{K-1+n} S_{\mathbf{a}}(K-1), \end{aligned}$$

where $S_{\mathbf{a}}(K) = \#\left\{\mathbf{y} \in \mathcal{O}^n : |\mathbf{y}| < |t|^K, \mathbf{y} \equiv \mathbf{a} \bmod t, F(\mathbf{y}) \equiv 0 \bmod t^K\right\}$. Similar to [40, p. 244], we have $S_{\mathbf{a}}(K) = q^{n-1} S_{\mathbf{a}}(K-1)$. Thus, for $K \geqslant 3$, $S_{t^K,t,\mathbf{a}}(\mathbf{0}) = 0$.

It remains to analyse $S_{\varpi^e}(\mathbf{0})$. By (3.7), we have $S_1(\mathbf{0}) = 1$. Moreover, by [15, (5.2), (5.3)], we have $S_{\varpi}(\mathbf{0}) \ll |\varpi|^{\frac{n}{2}+1}$ and $S_{\varpi^2}(\mathbf{0}) \ll |\varpi|^{n+2}$. Also, [15, Lemma 5.3] implies that

$S_{\varpi^3}(\mathbf{0}) \ll |\varpi|^{2n+3}$ and $S_{\varpi^4}(\mathbf{0}) \ll |\varpi|^{3n+3}$. By similar arguments as above, for $e \geqslant 5$ we have

$$S_{\varpi^e}(\mathbf{0}) = |\varpi|^e S_0(\varpi^e) - |\varpi|^{e-1+n} S_0(\varpi^{e-1}),$$

where $S_0(\varpi^e) = \#\{\mathbf{x} \in \mathcal{O}^n : |\mathbf{x}| < |\varpi|^e, F(\mathbf{x}) \equiv 0 \bmod \varpi^e\}$, as in the proof of Lemma 3.2.6. Then, by (3.11) – (3.14), it follows that for $e = 3k + l \geqslant 5$, where $l \in \{0,1,2\}$, we have

$$\begin{aligned} S_{\varpi^e}(\mathbf{0}) =& |\varpi|^{e+2n(k-1)} \begin{cases} \left(S_0(\varpi^3) - |\varpi|^{n-1} S_0(\varpi^2)\right), & \text{if } l = 0, \\ \left(S_0(\varpi^4) - |\varpi|^{n-1} S_0(\varpi^3)\right), & \text{if } l = 1, \\ \left(S_0(\varpi^5) - |\varpi|^{n-1} S_0(\varpi^4)\right), & \text{if } l = 2, \end{cases} \\ =& |\varpi|^{e+2nk} \begin{cases} 1 - |\varpi|^{-1}, & \text{if } e = 3k, \\ S_1(\varpi) + 1 - |\varpi|^{n-1}, & \text{if } e = 3k+1, \\ |\varpi|^n - |\varpi|^{n-1}, & \text{if } e = 3k+2. \end{cases} \end{aligned}$$

Thus, $|\varpi|^{-en} S_{\varpi^e}(\mathbf{0}) \ll |\varpi|^{-2n+5}$ for $e \geqslant 5$. Putting everything together, we obtain

$$\mathfrak{S} = q^{-n+1} \prod_{\substack{\varpi \text{ prime} \\ \varpi \neq t}} \left(1 + O(|\varpi|^{-\frac{n}{2}+1})\right),$$

where the implied constant is independent of q.

Then, there exists a constant c, independent of q, such that

$$\log q^{n-1}\mathfrak{S} = \sum_{\substack{\varpi \text{ prime} \\ \varpi \neq t}} \log\left(1 + \frac{c}{\varpi^{\frac{n}{2}-1}}\right) = \sum_{d \geqslant 1} \sum_{\substack{|\varpi| = q^d \\ \varpi \neq t \\ \varpi \text{ prime}}} \sum_{m \geqslant 1} \frac{1}{m} \left(\frac{c}{\varpi^{\frac{n}{2}-1}}\right)^m = O(q^{2-\frac{n}{2}}),$$

by the same argument as in (2.5). Since $\exp(z) = 1 + O(|z|)$, we have $q^{n-1}\mathfrak{S} = 1 + O(q^{2-\frac{n}{2}})$, which concludes the proof. □

Thus for $n \geqslant 10$, we have

$$M(d) = \mathfrak{S}\mathfrak{J}|P|^{n-3} + O(q^{11n/12+2/3}|P|^{3n/4-1}),$$

where $\mathfrak{S}$ and $\mathfrak{J}$ are given by Lemma 3.3.2 and (3.18), respectively. Note that the error term is satisfactory for Theorem 3.1.1.

3.4 ERROR TERM

There is a satisfactory contribution to $N(d)$ from $|\theta| < \widehat{Q}^{-5}$, since by (3.6), such terms contribute

$$< \sum_{\substack{r \text{ monic} \\ |r| \leqslant \widehat{Q}}} \widehat{Q} \cdot \widehat{Q}^{-5} |P|^n < \widehat{Q}^{-3} |P|^n < |P|^{n-9/2} < |P|^{n-3}.$$

Thus, we focus on the contribution from $|\theta| \geqslant \widehat{Q}^{-5}$. As in [15], let $Y, \Theta \in \mathbb{Z}$ be such that

$$0 \leqslant Y \leqslant Q, \quad -5Q \leqslant \Theta < -(Y+Q). \tag{3.20}$$

We will analyse the contribution to $N(d)$ coming from $\mathbf{c} \neq \mathbf{0}$ and r, θ such that $|r| = \widehat{Y}$ and $|\theta| = \widehat{\Theta}$. Denote this contribution by $E(d) = E(d; Y, \Theta)$. This section is similar to [15, Section 7.4] and [15, Section 8], however, we need to consider the cases when $t \mid r$ and $t \nmid r$ separately. Thus, let

$$B = \begin{cases} 0, & \text{if } t \mid r, \\ 1, & \text{if } t \nmid r. \end{cases}$$

Moreover, note that Lemma 3.2.2 imposes a constraint on $|\mathbf{c}|$. More precisely,

$$|\mathbf{c}| \leqslant \widehat{C} = q|\tilde{r}||P|^{-1} J(\Theta) = q^{B+1} |r||P|^{-1} J(\Theta),$$

where $\tilde{r} = rt^B$ and

$$J(\Theta) = \max\left\{1, \widehat{\Theta}|P|^3\right\}. \tag{3.21}$$

Then $E(d)$ is given by

$$\sum_{\substack{\mathbf{c} \in \mathcal{O}^n \\ 0 < |\mathbf{c}| \leqslant q^{B+1} \widehat{Y} |P|^{-1} J(\Theta)}} |P|^n \sum_{\substack{r \in \mathcal{O} \\ |r| = \widehat{Y} \\ r \text{ monic}}} q^{-Bn} |r|^{-n} \int_{|\theta| = \widehat{\Theta}} S_{r,t,\mathbf{a}}(\mathbf{c}) I_{\tilde{r}}(\theta; \mathbf{c}) \, d\theta.$$

By Lemma 3.2.7, $|I_{\tilde{r}}(\theta; \mathbf{c})| \ll L(\Theta)$, where

$$L(\Theta) = \min\left\{q^{-n}, q^n \widehat{\Theta}^{-\frac{n}{2}} |P|^{-\frac{3n}{2}}\right\}. \tag{3.22}$$

Thus, $E(d)$ is

$$\ll |P|^n \sum_{\substack{\mathbf{c} \in \mathcal{O}^n \\ \mathbf{c} \neq \mathbf{0} \\ |\mathbf{c}| \leqslant q^{B+1} \widehat{Y} |P|^{-1} J(\Theta)}} \sum_{\substack{r \in \mathcal{O} \\ |r| = \widehat{Y} \\ r \text{ monic}}} q^{-Bn} |r|^{-n} \int_{|\theta| = \widehat{\Theta}} |S_{r,t,\mathbf{a}}(\mathbf{c})| \, L(\Theta) \, d\theta. \tag{3.23}$$

Moreover, since we must have $\mathbf{c} \neq \mathbf{0}$ and $\mathbf{c} \in \mathcal{O}^n$, we get the following bound

$$\widehat{Y} \geqslant \frac{|P|}{J(\Theta) q^{B+1}}. \tag{3.24}$$

Let S be a set of finite primes to be decided upon in due course but which contains t. Any $r \in \mathcal{O}$ can be written as $r = b_1' b_1'' r_2$, where b_1' is square free such that $\varpi \mid b_1' \Rightarrow \varpi \in S$ and b_1'' is square-free and coprime to S. According to Lemma 3.25 there exist $M_1, M_2 \in \{1, t\}$ such that $M_1 \mid b_1'$ and $M_2 \mid r_2$, together with $\mathbf{b}_1, \mathbf{b}_2 \in (\mathcal{O}/t\mathcal{O})^n$ such that

$$|S_{r,t,\mathbf{a}}(\mathbf{c})| = |S_{b_1'',1,\mathbf{0}}(\mathbf{c}) S_{b_1',M_1,\mathbf{b}_1}(\mathbf{c}) S_{r_2,M_2,\mathbf{b}_2}(\mathbf{c})|. \tag{3.25}$$

Clearly, $t \nmid b_1''$ for any r and

$$\begin{cases} M_1 = t, M_2 = 1, & \text{if } t \mid\mid r, \\ M_1 = 1, M_2 = t, & \text{if } t^2 \mid r, \\ M_1 = 1, M_2 = 1, & \text{otherwise.} \end{cases}$$

Moreover, by [15, Lemma 2.2] we have

$$\int_{|\theta| = \widehat{\Theta}} \mathrm{d}\theta = \widehat{\Theta + 1} - \widehat{\Theta} \leqslant \widehat{\Theta + 1}.$$

Let $\mathcal{O}^\sharp = \{b \in \mathcal{O} : b \text{ is monic and square-free}\}$. There exist $\mathbf{b}_1, \mathbf{b}_2 \in (\mathcal{O}/t\mathcal{O})^n$ such that the bound for $E(d)$ in (3.23) becomes

$$\ll \sum_{\substack{\mathbf{c} \in \mathcal{O}^n \\ \mathbf{c} \neq \mathbf{0} \\ |\mathbf{c}| \leqslant q^{B+1} \widehat{Y} |P|^{-1} J(\Theta)}} \frac{|P|^n \widehat{\Theta + 1} L(\Theta)}{q^{Bn} \widehat{Y}^{\frac{n-1}{2}}} \sum_{\substack{r_2 \in \mathcal{O} \\ |r_2| \leqslant \frac{\widehat{Y}}{|b_1'' b_1'|}}} \sum_{\substack{b_1' \in \mathcal{O}^\sharp \\ \varpi | b_1' \Rightarrow \varpi \in S}} \frac{\left| S_{b_1', M_1, \mathbf{b}_1}(\mathbf{c}) S_{r_2, M_2, \mathbf{b}_2}(\mathbf{c}) \right|}{|b_1' r_2|^{\frac{n+1}{2}}} \cdot \mathcal{S},$$

where

$$\mathcal{S} = \sum_{\substack{b_1'' \in \mathcal{O}^\sharp \\ (b_1'', S) = 1 \\ |b_1' b_1'' r_2| = \widehat{Y}}} \frac{\left| S_{b_1'', 1, \mathbf{0}}(\mathbf{c}) \right|}{|b_1''|^{\frac{n+1}{2}}}.$$

From now put $b = b_1''$, and $d = b_1' r_2$, for simplicity. Also, write $S_b(\mathbf{c}) = S_{b,1,\mathbf{0}}(\mathbf{c})$. Moreover, take

$$S = \begin{cases} \{\varpi : \varpi \mid tF^*(\mathbf{c})\}, & \text{if } F^*(\mathbf{c}) \neq 0, \\ \{\varpi : \varpi \mid t\}, & \text{otherwise.} \end{cases}$$

Lemma 3.4.1. *We have*

$$\mathcal{S} \ll \begin{cases} \widehat{Y}^{\varepsilon}\frac{\widehat{Y}}{|d|}, & \text{if } F^*(\mathbf{c}) \neq 0, \\ \widehat{Y}^{\varepsilon}\left(\frac{\widehat{Y}}{|d|}\right)^{1+1/2}, & \text{otherwise}, \end{cases}$$

where F^* is the dual form of F.

Proof. Recall that in our case $|\Delta_F| = 1$. Furthermore, by [40, Lemma 12] and [43, Lemma 60], there exists a constant $A(n) > 0$ depending only on n such that for a prime ϖ we have

$$S_{\varpi}(\mathbf{c}) \leqslant A(n)|\varpi|^{\frac{n+1}{2}}\left|(\varpi, F^*(\mathbf{c}))\right|^{1/2}. \tag{3.26}$$

By Lemma 3.25, $S_b(\mathbf{c})$ is a multiplicative function of b. Thus, by (3.26) and Lemma 2.2.5 we have

$$\mathcal{S} = \sum_{\substack{b\in\mathcal{O}^\sharp \\ (b,S)=1 \\ |bd|=\widehat{Y}}} \frac{|S_b(\mathbf{c})|}{|b|^{\frac{n+1}{2}}} \ll \sum_{\substack{b\in\mathcal{O}^\sharp \\ (b,S)=1 \\ |bd|=\widehat{Y}}} \left|(b, F^*(\mathbf{c}))\right|^{1/2}|b|^{\varepsilon} \ll \widehat{Y}^{\varepsilon}\sum_{\substack{b\in\mathcal{O}^\sharp \\ (b,S)=1 \\ |bd|=\widehat{Y}}} \left|(b, F^*(\mathbf{c}))\right|^{1/2}.$$

The definition of S and the constraint that $(b, S) = 1$ imply that

$$(b, F^*(\mathbf{c})) = \begin{cases} 1, & \text{if } F^*(\mathbf{c}) \neq 0, \\ b, & \text{otherwise}, \end{cases}$$

and this concludes the proof. □

Thus, we will consider separately the cases $F^*(\mathbf{c}) \neq 0$ and $F^*(\mathbf{c}) = 0$ separately. Let $E_1(d)$ and $E_2(d)$ denote the contributions to $E(d)$ coming from $\mathbf{c}$ such that $F^*(\mathbf{c}) \neq 0$ and $F^*(\mathbf{c}) = 0$, respectively.

3.4.1 *Treatment of the generic term*

Suppose $F^*(\mathbf{c}) \neq 0$. Then, by the first part of Lemma 3.4.1 we have

$$E_1(d) \ll \frac{|P|^n\widehat{\Theta+1}L(\Theta)\widehat{Y}^{\frac{3-n}{2}+\varepsilon}}{q^{Bn}} \sum_{\substack{\mathbf{c}\in\mathcal{O}^n \\ \mathbf{c}\neq\mathbf{0} \\ |c|\leqslant q^{B+1}\widehat{Y}|P|^{-1}J(\Theta)}} R_1(\mathbf{c}),$$

where

$$R_1(\mathbf{c}) = \sum_{\substack{r_2\in\mathcal{O} \\ |r_2|\leqslant\frac{\widehat{Y}}{|b_1'|}}} \sum_{\substack{b_1'\in\mathcal{O}^\sharp \\ \varpi|b_1'\Rightarrow\varpi\in S}} \frac{\left|S_{b_1',M_1,\mathbf{b}_1}(\mathbf{c})S_{r_2,M_2,\mathbf{b}_2}(\mathbf{c})\right|}{|b_1'r_2|^{\frac{n+3}{2}}}.$$

Then, (3.26) implies that

$$R_1(\mathbf{c}) \ll \sum_{\substack{b_1' \in \mathcal{O}^\sharp \\ \varpi | b_1' \Rightarrow \varpi \in S}} \frac{1}{|b_1'|^{1-\varepsilon}} \sum_{\substack{r_2 \in \mathcal{O} \\ |r_2| \leqslant \widehat{Y}}} \frac{|S_{r_2, M_2, \mathbf{b}_2}(\mathbf{c})|}{|r_2|^{\frac{n+3}{2}}} \ll \sum_{\substack{r_2 \in \mathcal{O} \\ |r_2| \leqslant \widehat{Y}}} \frac{|S_{r_2, M_2, \mathbf{b}_2}(\mathbf{c})|}{|r_2|^{\frac{n+3}{2}}},$$

since by Lemma 2.2.5 we have

$$\sum_{\substack{b_1' \in \mathcal{O}^\sharp \\ \varpi | b_1' \Rightarrow \varpi \in S}} \frac{1}{|b_1'|^{1-\varepsilon}} = \prod_{\varpi \in S} \left(1 - \frac{1}{|\varpi|^{1-\varepsilon}}\right)^{-1} = \prod_{\varpi \in S} \sum_{k=0}^{\infty} \frac{1}{|\varpi|^{k(1-\varepsilon)}} \leqslant \prod_{\varpi \in S} C_\varepsilon \ll |P|^\varepsilon.$$

Decompose r_2 as $b_2 r_3$. Then, by the multiplicativity property in Lemma 3.25, we have $|S_{r_2,M_2,\mathbf{b}_2}(\mathbf{c})| \leqslant \left|S_{b_2,M_2',\mathbf{b}_2'}(\mathbf{c}) S_{r_3,M_3,\mathbf{b}_3}(\mathbf{c})\right|$, for appropriate $M_2', M_3 \in \{1, t\}$ and $\mathbf{b}_2', \mathbf{b}_3 \in (\mathcal{O}/t\mathcal{O})^n$. Thus,

$$\sum_{\substack{r_2 \in \mathcal{O} \\ |b_1' r_2| \leqslant \widehat{Y}}} \frac{|S_{r_2, M_2, \mathbf{b}_2}(\mathbf{c})|}{|r_2|^{\frac{n+3}{2}}} \leqslant \sum_{\substack{b_2 r_3 \in \mathcal{O} \\ |b_1' b_2 r_3| \leqslant \widehat{Y}}} \frac{\left|S_{b_2,M_2',\mathbf{b}_2'}(\mathbf{c}) S_{r_3,M_3,\mathbf{b}_3}(\mathbf{c})\right|}{|b_2 r_3|^{\frac{n+3}{2}}}.$$

Moreover, applying Lemma 3.2.5 with $|M_3| \leqslant q$,

$$\sum_{\substack{\mathbf{c} \in \mathcal{O}^n \\ \mathbf{c} \neq \mathbf{0} \\ |\mathbf{c}| \leqslant \widehat{C}}} \sum_{\substack{r_3 \in \mathcal{O} \\ |r_3| \leqslant \frac{\widehat{Y}}{|b_2|}}} \frac{|S_{r_3, M_3, \mathbf{b}_3}(\mathbf{c})|}{|r_3|^{\frac{n+3}{2}}} \ll q^n |P|^\varepsilon \left(\widehat{Y}^{n/3-1/6} + \widehat{C}^n\right). \tag{3.27}$$

Lemma 3.4.2. *For M_2' and $\mathbf{b}_2'$ as above and $Y \in \mathbb{Z}$, there exists some $\varepsilon > 0$ such that*

$$\sum_{\substack{b_2 \in \mathcal{O} \\ |b_2| \leqslant \widehat{Y}}} \frac{\left|S_{b_2, M_2', \mathbf{b}_2'}(\mathbf{c})\right|}{|b_2|^{\frac{n+3}{2}}} \ll |P|^\varepsilon,$$

for any $\mathbf{c} \in \mathcal{O}^n$.

Proof. Suppose $M_2' = 1$ so that $S_{b_2,M_2',\mathbf{b}_2'}(\mathbf{c}) = S_{b_2,1,\mathbf{0}}(\mathbf{c})$. By [15, (5.3)], together with the fact that in our case $|\Delta_F| = 1$, there exists a constant $A(n) > 0$ depending only on n such that

$$S_{\varpi^2}(\mathbf{c}) \leqslant A(n) |\varpi|^{n+1} \left|(\varpi, F^*(\mathbf{c}))\right|. \tag{3.28}$$

Then, by the multiplicativity property in Lemma 3.25 and by (3.28),

$$\sum_{\substack{b_2 \in \mathcal{O} \\ |b_2| \leqslant \widehat{Y}}} \frac{|S_{b_2,1,\mathbf{0}}(\mathbf{c})|}{|b_2|^{\frac{n+3}{2}}} = \sum_{\substack{k_2 \in \mathcal{O} \\ |k_2|^2 \leqslant \widehat{Y}}} \frac{\left|S_{k_2^2,1,\mathbf{0}}(\mathbf{c})\right|}{|k_2|^{n+3}} \ll \sum_{\substack{k_2 \in \mathcal{O} \\ |k_2| \leqslant \widehat{Y}^{1/2}}} \frac{|A(n)|^{\omega(k_2)} |k_2|^{n+1} \left|(k_2, F^*(\mathbf{c}))\right|}{|k_2|^{n+3}}.$$

It follows from Lemma 2.2.5 that this can be bounded by

$$\ll |P|^{\varepsilon} \sum_{\substack{k_2 \in \mathcal{O} \\ |k_2| \leqslant \widehat{Y}^{1/2}}} \frac{|(k_2, F^*(\mathbf{c}))|}{|k_2|^2} \ll |P|^{\varepsilon} \sum_{\substack{k_2 \in \mathcal{O} \\ |k_2| \leqslant \widehat{Y}^{1/2}}} \frac{1}{|k_2|} \ll |P|^{\varepsilon},$$

where the last inequality follows from Lemma 3.2.1 since k_2 is monic.

We now consider the case $M_2' = t$. First, we need to bound the sum

$$S_{t^2,t,\mathbf{b}_2'}(\mathbf{c}) = \sideset{}{^*}\sum_{|a|<|t|^2} \sum_{\substack{\mathbf{y} \in \mathcal{O}^n \\ |\mathbf{y}|<|t|^2 \\ \mathbf{y} \equiv \mathbf{b}_2' \bmod t}} \psi\left(\frac{aF(\mathbf{y}) - \mathbf{c}.\mathbf{y}}{t^2}\right).$$

Making a change of variables $\mathbf{y} = \mathbf{b}_2' + t\mathbf{z}$, we get

$$S_{t^2,t,\mathbf{b}_2'}(\mathbf{c}) = \psi\left(\frac{-\mathbf{c}.\mathbf{b}_2'}{t^2}\right) \sideset{}{^*}\sum_{|a|<|t|^2} \psi\left(\frac{aF(\mathbf{b}_2')}{t^2}\right) \sum_{\substack{\mathbf{z} \in \mathcal{O}^n \\ |\mathbf{z}|<|t|}} \psi\left(\frac{a\mathbf{z}.\nabla F(\mathbf{b}) - \mathbf{c}.\mathbf{z}}{t}\right).$$

But Lemma 2.1.2 implies that

$$\sum_{\substack{\mathbf{z} \in \mathcal{O}^n \\ |\mathbf{z}|<|t|}} \psi\left(\frac{a\mathbf{z}.\nabla F(\mathbf{b}) - \mathbf{c}.\mathbf{z}}{t}\right) = \begin{cases} q^n, & \text{if } |a\nabla F(\mathbf{b}) - \mathbf{c}| < 1, \\ 0, & \text{otherwise,} \end{cases}$$

and hence

$$|S_{t^2,t,\mathbf{b}_2'}(\mathbf{c})| \leqslant q^n \left|\{a \in \mathcal{O} : |a| < |t|^2 : |a\nabla F(\mathbf{b}) - \mathbf{c}| < 1\}\right| \leqslant q^n. \tag{3.29}$$

By the definition of $S_{b_2,M_2',\mathbf{b}_2'}$ and the multiplicativity property in Lemma 3.25, we can write it as $S_{b_2,t,\mathbf{b}_2'}(\mathbf{c}) = S_{t^2,t,\mathbf{b}_2'}(\mathbf{c}) S_{(k_2/t)^2,1,\mathbf{0}}(\mathbf{c})$. The second sum is well understood and can be bounded using (3.28), giving

$$\left|S_{(k_2/t)^2,1,\mathbf{0}}(\mathbf{c})\right| \ll |P|^{\varepsilon} |b_2|^{\frac{n+2}{2}} \frac{1}{q^{n+2}}.$$

Then, by (3.29) we have $\left|S_{b_2,t,\mathbf{b}_2'}(\mathbf{c})\right| \ll q^{-2}|P|^{\varepsilon}|b_2|^{\frac{n+2}{2}}$. Hence,

$$\sum_{\substack{b_2 \in \mathcal{O} \\ |b_2| \leqslant \widehat{Y}}} \frac{\left|S_{b_2,t,\mathbf{b}_2'}(\mathbf{c})\right|}{|b_2|^{\frac{n+3}{2}}} \ll q^{-2}|P|^{\varepsilon} \sum_{\substack{b_2 \in \mathcal{O} \\ |b_2| \leqslant \widehat{Y}}} \frac{1}{|b_2|^{\frac{1}{2}}} = q^{-2}|P|^{\varepsilon} \sum_{\substack{k_2 \in \mathcal{O} \\ |k_2| \leqslant \widehat{Y}^{1/2}}} \frac{1}{|k_2|} \ll q^{-2}|P|^{\varepsilon},$$

by Lemma 3.2.1. □

Thus, putting everything back together,

$$\sum_{\substack{\mathbf{c} \in \mathcal{O}^n \\ \mathbf{c} \neq \mathbf{0} \\ |\mathbf{c}| \leqslant q^{B+1}\widehat{Y}J(\Theta)/|P|}} R_1(\mathbf{c}) \ll q^n |P|^{\varepsilon} \left(\widehat{Y}^{n/3-1/6} + \left(q^{B+1}\widehat{Y}|P|^{-1}J(\Theta)\right)^n\right),$$

and hence,

$$E_1(d) \ll q^n |P|^{\varepsilon} \widehat{\Theta + 1} L(\Theta) \left(\frac{|P|^n}{q^{Bn} \widehat{Y}^{\frac{n}{6} - \frac{4}{3}}} + q^n J(\Theta)^n \widehat{Y}^{\frac{n+3}{2}} \right).$$

Then, by (3.22) and (3.24), the first term is

$$\ll q^{\frac{7n}{6} - \frac{1}{3} - B(\frac{5n}{6} + \frac{4}{3})} \widehat{\Theta} |P|^{\frac{5n}{6} + \frac{4}{3} + \varepsilon} J(\Theta)^{\frac{n}{6} - \frac{4}{3}} \min \left\{ q^{-n}, q^n \widehat{\Theta}^{-\frac{n}{2}} |P|^{-\frac{3n}{2}} \right\}.$$

Noting that $B \in \{0, 1\}$ and $\min\{X, Z\} \leqslant X^u Z^v$ for any $u, v \geqslant 0$ such that $u + v = 1$, by (3.21), we obtain

$$\ll q^{\frac{7n}{6} - \frac{1}{3} - nu + nv} \widehat{\Theta}^{1 - \frac{nv}{2}} |P|^{\frac{5n}{6} + \frac{4}{3} - \frac{3nv}{2} + \varepsilon} \max \left\{ 1, \widehat{\Theta} |P|^3 \right\}^{\frac{n}{6} - \frac{4}{3}}.$$

If $\widehat{\Theta}|P|^3 \leqslant 1$, take $u = 1 - \frac{2}{n}$ and $v = \frac{2}{n}$. Then, we obtain $\ll q^{\frac{n}{6} + \frac{11}{3}} |P|^{\frac{5n}{6} - \frac{5}{3} + \varepsilon}$. Otherwise, if $\widehat{\Theta}|P|^3 > 1$, take $u = \frac{2}{3} + \frac{2}{3n}$ and $v = \frac{1}{3} - \frac{2}{3n}$. Then, we get $\ll q^{\frac{5n}{6} - \frac{5}{3}} |P|^{\frac{5n}{6} - \frac{5}{3} + \varepsilon}$.

Similarly, by (3.21) and (3.22), the second term is

$$\ll q^{2n + 1 - nu + nv} \widehat{\Theta}^{1 - \frac{nv}{2}} \max \left\{ 1, \widehat{\Theta} |P|^3 \right\}^n |P|^{-\frac{3nv}{2} + \varepsilon} \widehat{Y}^{\frac{n+3}{2}},$$

for any $u, v \geqslant 0$ such that $u + v = 1$. If $\widehat{\Theta}|P|^3 \leqslant 1$, then by (3.20), we have

$$\ll q^{2n + 1 - nu + nv} \widehat{\Theta}^{1 - \frac{nv}{2}} |P|^{\frac{3n}{4} + \frac{9}{4} - \frac{3nv}{2} + \varepsilon}.$$

Taking $u = 1 - \frac{2}{n}$ and $v = \frac{2}{n}$, we get $\ll q^{n+5} |P|^{\frac{3n}{4} - \frac{3}{4} + \varepsilon}$. Otherwise, if $\widehat{\Theta}|P|^3 > 1$, we have

$$\ll q^{2n + 1 - nu + nv} \widehat{\Theta}^{1 + n - \frac{nv}{2}} |P|^{3n - \frac{3nv}{2} + \varepsilon} \widehat{Y}^{\frac{n+3}{2}}.$$

On noting that the exponent of $\widehat{\Theta}$ is strictly positive for any $v \leqslant 1$, by (3.20), we have

$$\ll q^{2n + 1 - nu + nv} |P|^{\frac{3n}{2} - \frac{3nv}{4} - \frac{3}{2} + \varepsilon} \widehat{Y}^{-\frac{n}{2} + \frac{1}{2} + \frac{nv}{2}}.$$

Taking $u = \frac{1}{n}$ and $v = 1 - \frac{1}{n}$, we get $\ll q^{3n-1} |P|^{\frac{3n}{4} - \frac{3}{4} + \varepsilon}$.

Thus, for $n \geqslant 10$ we have $E_1(d) \ll q^{\frac{5n}{6} - \frac{5}{3}} |P|^{\frac{5n}{6} - \frac{5}{3} + \varepsilon} + q^{3n-1} |P|^{\frac{3n}{4} - \frac{3}{4} + \varepsilon}$. Hence,

$$E_1(d) \ll |P|^{\varepsilon} \left(q^{\frac{5(d+2)n}{6} - \frac{5(d+2)}{3}} + q^{\frac{3(d+5)n}{4} - \frac{3d+7}{4}} \right),$$

which is satisfactory for Theorem 3.1.1.

3.4.2 *Treatment of* $E_2(d)$

Suppose $F^*(\mathbf{c}) = 0$. Denote the contribution to $E(d)$ coming from $\mathbf{c}$ such that $F^*(\mathbf{c}) = 0$ by $E_2(d)$. Then, by the second part of Lemma 3.4.1, we have

$$E_2(d) \ll \frac{|P|^n \widehat{\Theta + 1} \widehat{Y}^{2-\frac{n}{2}+\varepsilon} L(\Theta)}{q^{Bn}} \sum_{\substack{\mathbf{c}\in\mathcal{O}^n \\ \mathbf{c}\neq\mathbf{0} \\ |\mathbf{c}|\ll q^{B+1}\widehat{Y}|P|^{-1}J(\Theta)}} R_2(\mathbf{c}),$$

where

$$R_2(\mathbf{c}) = \sum_{\substack{r_2\in\mathcal{O} \\ |r_2|\leqslant \frac{\widehat{Y}}{|b_1'|}}} \sum_{\substack{b_1'\in\mathcal{O}^\sharp \\ \varpi|b_1'\Rightarrow\varpi\in S}} \frac{\left|S_{b_1',M_1,\mathbf{b}_1}(\mathbf{c}) S_{r_2,M_2,\mathbf{b}_2}(\mathbf{c})\right|}{|b_1' r_2|^{\frac{n}{2}+2}}.$$

As in Section 3.4.1, decompose r_2 as $b_2 r_3$. Then, using Lemma 3.2.6 and $|M_3| \leqslant q$, we get

$$\sum_{\substack{\mathbf{c}\in\mathcal{O}^n \\ \mathbf{c}\neq\mathbf{0} \\ |\mathbf{c}|\leqslant\widehat{C} \\ F^*(\mathbf{c})=0}} \sum_{\substack{r_3\in\mathcal{O} \\ |r_3|=\widehat{R}}} \frac{|S_{r_3,M_3,\mathbf{b}_3}(\mathbf{c})|}{|r_3|^{\frac{n}{2}+2}} \ll q^{\frac{2n-5}{3}+\varepsilon} \widehat{R}^{\frac{n}{3}-\frac{1}{2}+\varepsilon} \left(1+\widehat{C}\right)^{n-\frac{3}{2}+\varepsilon}.$$

Then,

$$\sum_{\substack{\mathbf{c}\in\mathcal{O}^n \\ \mathbf{c}\neq\mathbf{0} \\ |\mathbf{c}|\leqslant\widehat{C} \\ F^*(\mathbf{c})=0}} \sum_{\substack{r_3\in\mathcal{O} \\ |r_3|\leqslant \frac{\widehat{Y}}{|b_1' b_2|}}} \frac{|S_{r_3,M_3,\mathbf{b}_3}(\mathbf{c})|}{|r_3|^{\frac{n}{2}+2}} \ll q^{\frac{2n-5}{3}+\varepsilon} \widehat{Y}^{\frac{n}{3}-\frac{1}{2}+\varepsilon} \left(1+\widehat{C}^{n-\frac{3}{2}+\varepsilon}\right) \tag{3.30}$$

Moreover, Lemma 3.4.2 implies that

$$\sum_{\substack{b_2\in\mathcal{O} \\ |b_2|\leqslant\widehat{Y}}} \frac{\left|S_{b_2,M_2',\mathbf{b}_2'}(\mathbf{c})\right|}{|b_2|^{\frac{n}{2}+2}} \ll |P|^\varepsilon. \tag{3.31}$$

It follows from (3.26) that

$$R_2(\mathbf{c}) \ll \sum_{\substack{b_1'\in\mathcal{O}^\sharp \\ \varpi|b_1'\Rightarrow\varpi\in S}} |b_1'|^{\varepsilon-1} \sum_{\substack{r_2\in\mathcal{O} \\ |b_1' r_2|\leqslant\widehat{Y}}} \frac{|S_{r_2,M_2,\mathbf{b}_2}(\mathbf{c})|}{|r_2|^{\frac{n}{2}+2}}.$$

Then, by (3.30) and (3.31), we have

$$\sum_{\substack{\mathbf{c}\in\mathcal{O}^n \\ \mathbf{c}\neq\mathbf{0} \\ |\mathbf{c}|\leqslant q^{B+1}\widehat{Y}|P|^{-1}J(\Theta)}} R_2(\mathbf{c}) \ll q^{\frac{2n}{3}-\frac{5}{3}} \widehat{Y}^{\frac{n}{3}-\frac{1}{2}+\varepsilon} \left(1+\left(q^{B+1}\widehat{Y}|P|^{-1}J(\Theta)\right)^{n-\frac{3}{2}}\right).$$

Thus, we can bound $E_2(d)$ by

$$q^{\frac{2n}{3}-\frac{2}{3}}|P|^{\varepsilon}\widehat{\Theta}\left(\frac{|P|^n L(\Theta)}{q^{Bn}\widehat{Y}^{\frac{n}{6}-\frac{3}{2}}}+q^{n-\frac{3(B+1)}{2}}|P|^{\frac{3}{2}}J(\Theta)^{n-\frac{3}{2}}L(\Theta)\widehat{Y}^{\frac{n+1}{2}}\right).$$

Then, by (3.22) and (3.24), the first term is

$$\ll q^{\frac{5n}{6}+\frac{5}{6}-B(\frac{5n}{6}-\frac{3}{2})}\widehat{\Theta}\min\left\{q^{-n},q^n\widehat{\Theta}^{-\frac{n}{2}}|P|^{-\frac{3n}{2}}\right\}J(\Theta)^{\frac{n}{6}-\frac{3}{2}}|P|^{\frac{5n}{6}+\frac{3}{2}+\varepsilon}.$$

Since $B\in\{0,1\}$ and $\min\{X,Z\}\leqslant X^uZ^v$ for any $u,v\geqslant 0$ such that $u+v=1$, by (3.21), we obtain

$$\ll q^{\frac{5n}{6}+\frac{5}{6}-nu+nv}\widehat{\Theta}^{1-\frac{nv}{2}}\max\left\{1,\widehat{\Theta}|P|^3\right\}^{\frac{n}{6}-\frac{3}{2}}|P|^{\frac{5n}{6}+\frac{3}{2}-\frac{3nv}{2}+\varepsilon}.$$

If $\widehat{\Theta}|P|^3\leqslant 1$, take $u=1-\frac{2}{n}$ and $v=\frac{2}{n}$. Then, we obtain $\ll q^{-\frac{n}{6}-\frac{5}{6}}|P|^{\frac{5n}{6}-\frac{3}{2}+\varepsilon}$. Otherwise, if $\widehat{\Theta}|P|^3>1$, take $u=\frac{2}{3}+\frac{1}{n}$ and $v=\frac{1}{3}-\frac{1}{n}$. Then, we get $\ll q^{\frac{n}{2}-\frac{7}{6}}|P|^{\frac{5n}{6}-\frac{3}{2}+\varepsilon}$.

Similarly, by (3.21), (3.22) and (3.24), the second term is

$$\ll q^{\frac{5n}{3}-\frac{13}{6}-nu+nv}\widehat{\Theta}^{1-\frac{nv}{2}}|P|^{\frac{3}{2}-\frac{3nv}{2}+\varepsilon}\max\left\{1,\widehat{\Theta}|P|^3\right\}^{n-\frac{3}{2}}\widehat{Y}^{\frac{n+1}{2}},$$

for any $u,v\geqslant 0$ such that $u+v=1$. If $\widehat{\Theta}|P|^3\leqslant 1$, then by (3.20), we have

$$\ll q^{\frac{5n}{3}-\frac{13}{6}-nu+nv}\widehat{\Theta}^{1-\frac{nv}{2}}|P|^{\frac{3n}{4}+\frac{9}{4}-\frac{3nv}{2}+\varepsilon}.$$

Taking $u=1-\frac{2}{n}$ and $v=\frac{2}{n}$, we get $\ll q^{\frac{2n}{3}+\frac{11}{6}}|P|^{\frac{3n}{4}-\frac{3}{4}+\varepsilon}$. Otherwise, if $\widehat{\Theta}|P|^3>1$, we have

$$\ll q^{\frac{5n}{3}-\frac{13}{6}-nu+nv}\widehat{\Theta}^{n-\frac{1}{2}-\frac{nv}{2}}|P|^{3n-3-\frac{3nv}{2}+\varepsilon}\widehat{Y}^{\frac{n+1}{2}}.$$

Since the exponent of $\widehat{\Theta}$ is strictly positive for any $v\leqslant 1$, by (3.20), we have

$$\ll q^{\frac{5n}{3}-\frac{13}{6}-nu+nv}\widehat{Y}^{-\frac{n}{2}+1+\frac{nv}{2}}|P|^{\frac{3n}{2}-\frac{9}{4}-\frac{3nv}{4}+\varepsilon}.$$

Taking $u=\frac{2}{n}$ and $v=1-\frac{2}{n}$, we get $\ll q^{\frac{8n}{3}-\frac{37}{6}}|P|^{\frac{3n}{4}-\frac{3}{4}+\varepsilon}$.

Thus, for $n\geqslant 10$ we have $E_2(d)\ll q^{\frac{n}{2}-\frac{7}{6}}|P|^{\frac{5n}{6}-\frac{3}{2}+\varepsilon}+q^{\frac{8n}{3}-\frac{37}{6}}|P|^{\frac{3n}{4}-\frac{3}{4}+\varepsilon}$. Hence,

$$E_2(d)\ll|P|^{\varepsilon}\left(q^{\frac{(5d+8)n}{6}-\frac{3d}{2}-\frac{8}{3}}+q^{\frac{(3d+11)n}{4}-\frac{3d}{4}-\frac{83}{12}}\right),$$

which is satisfactory for Theorem 3.1.1.

4

MANIN'S CONJECTURE FOR $\mathrm{Hilb}^2\, \mathbb{P}^2$

4.1 INTRODUCTION

Let X be a smooth projective variety defined over a global field K such that the set $X(K)$ of rational points is Zariski dense in X and let $\mathcal{L}$ be an ample line bundle in X, i.e. a line bundle $\mathcal{L}$ such that a positive power of $\mathcal{L}$ is very ample (see Section 2.4). One can define a height function $H_{\mathcal{L}}$ on $X(K)$ taking values in $\mathbb{R}_{>0}$ in order to study the asymptotic behaviour of the number of K-points of bounded height on X.

As described in Section 2.4, Manin and his collaborators (see [3], [27]) predict that when X is a Fano variety and K is a number field, there exists an open subset U of X such that

$$\#\left\{x \in U(K) : H_{\omega_X^{-1}}(x) \leqslant B\right\} \sim C_{H_{\omega_X^{-1}}} B (\log B)^{t-1},$$

as $B \to \infty$, where $H_{\omega_X^{-1}}$ is the height associated to the anticanonical line bundle ω_X^{-1} and t is the rank of the Picard group of X. Moreover, Peyre [65] gave a conjectural interpretation of the constant $C_{H_{\omega_X^{-1}}}$. The first counterexample was provided by Batyrev and Tschinkel [4]. This led to a refined version of Manin's conjecture in which one is allowed to remove a finite number of thin sets as defined by Serre [75, §3.1]. Results that support the need for the thin set version of this conjecture have been proven by Le Rudulier [58, Theorem 4.2] and Browning–Heath-Brown [14, Theorem 1.1]. We recall the definition of thin sets, but for more details we refer the reader to Section 2.4.

Definition 4.1.1. Let V be an irreducible algebraic variety defined over a field K. A *thin* set T is a set that is contained in a finite union of sets of the type (C_1) or (C_2), where a subset $A \subseteq V(K)$ is of type (C_1) if there exists a proper closed subset W of V such that $A \subset W(K)$, and of type (C_2) if there exists an irreducible variety W of V of dimension equal to $\dim V$ and a generically surjective morphism $\pi : W \to V$ of degree $\geqslant 2$ with $A \subset \pi(W(K))$.

In the case when K is a global field of positive characteristic, Peyre [67, Theorem 3.5.1] proposed an analogue of [65]. It is natural to extend the thin set version of Manin's conjecture to this case. The goal of this paper is to provide an example that supports this conjecture in the case of function fields of positive characteristic. Consider $\operatorname{Hilb}^2 \mathbb{P}^2$ over a function field K of degree e over $\mathbb{F}_q(t)$ such that $\operatorname{char}(K) > 2$.

Theorem 4.1.2. *There exists a non-empty thin set $Z_0 \subset \operatorname{Hilb}^2 \mathbb{P}^2(K)$ such that*

$$\#\left\{z \in \operatorname{Hilb}^2 \mathbb{P}^2(K) \backslash Z_0 : H_{\omega_X^{-1}}(z) = q^M\right\} = cq^M M + O\left(\sqrt{M} q^M\right),$$

as $M \to \infty$, where the leading constant agrees with the prediction of Peyre.

We shall present the geometry of the Hilbert scheme in Section 4.2, where we explain that the anticanonical divisor of $\operatorname{Hilb}^2 \mathbb{P}^2$ is big. Big divisors belong to the interior of the pseudo-effective cone and can be seen as a generalisation of ample divisors. We remark that Manin's conjecture together with Peyre's refinements have the same formulation in the case of varieties with big anticanonical divisor. We refer the reader to [57] for more geometric details. We shall also introduce a more refined version of Theorem 4.1.2, which can be seen as a function field analogue of [58, Theorem 4.2]. However, the result of Le Rudulier is only valid over $\mathbb{Q}$ and has not been proven over number fields, whereas Theorem 4.1.2 holds over any finite extension of $\mathbb{F}_q(t)$. Le Rudulier's strategy is to use the geometry of $\operatorname{Hilb}^2 \mathbb{P}^2$ to reduce the problem to counting points in $\mathbb{P}^2$ that are quadratic over $\mathbb{Q}$. Then, one appeals to work of Schmidt [73, Theorem 3] which we introduced in Section 2.5. Suppose K is a number field of degree e over $\mathbb{Q}$ and d is a positive integer. Let us define the quantity

$$N_K(n+1, d, B) = \#\left\{P \in \mathbb{P}^n(\overline{\mathbb{Q}}) : H'(x) \leqslant B, [K(P) : K] \leqslant d\right\},$$

where $H'(x) = \prod_{v \in \Omega_K} \max_{0 \leqslant i \leqslant n} |x_i|_v$. Schanuel [71] proved that

$$N_K(n+1, 1, B) = S_K(n+1, 1) B^{(n+1)e} + O_{K,n}\left(B^{(n+1)e-1}\right), \tag{4.1}$$

with an additional $\log B$ term in the error when $K = \mathbb{Q}$ and $n = 1$. The leading constant is called the *Schanuel constant*

$$S_K(n+1, 1) = \left(\frac{2^r (2\pi)^s}{|\Delta|^{1/2}}\right)^{n+1} \frac{(n+1)^{r+s-1} hR}{w \zeta_K(n+1)}, \tag{4.2}$$

where r is the number of real embeddings of K, s the number of pairs of distinct complex conjugate embeddings of K, Δ the discriminant of K, h is the class number of K, R the regulator of K, w the number of roots of unity in K, and ζ_K the Dedekind zeta-function of K. Schmidt [73] generalised this result to quadratic number fields. More precisely, the result we are interested in states that

$$N_{\mathbb{Q}}(3,2,B) = \frac{24+2\pi^2}{\zeta(3)^2} B^6 \log B + O\left(B^6\sqrt{\log B}\right), \tag{4.3}$$

where the leading constant is a sum of Schanuel constants over extensions of degree $d = 2$ over $K = \mathbb{Q}$. Note that the asymptotic formula in [73, Theorem 3] is

$$\frac{48+4\pi^2}{\zeta(3)^2} B^3 \log B + O\left(B^3\sqrt{\log B}\right),$$

due to the fact that Schmidt uses a second power of the absolute height. For more details one how the choice of height affects the asymptotic formula we refer the reader to Section 2.5.1.

The method of proof for Theorem 4.1.2 is similar, and begins with a function field analogue of (4.3). Given a function field K, let $e = [K : \mathbb{F}_q(t)]$ and define the quantity

$$N_K(n+1,d,M) = \#\left\{P \in \mathbb{P}^n(\overline{\mathbb{F}_q(t)}) : H_n(P) = q^{\frac{M}{ed}}, [K(P):K] = d\right\},$$

where H_n denotes the usual absolute height on projective space $\mathbb{P}^n$ with respect to the ground field $\mathbb{F}_q(t)$. We note that e depends not just on K, but also on the choice of genus 0 subfield $\mathbb{F}_q(t)$ of K. More precisely, given a point $x \in \mathbb{P}^n(\overline{\mathbb{F}_q(t)})$ of degree d over K with homogeneous coordinates $[x_0 : \ldots : x_n]$ we have

$$H_n(x) = \left(\prod_{v\in\Omega_{K(x)}} \max_{0\leqslant i\leqslant n} |x_i|_v\right)^{\frac{1}{de}}, \tag{4.4}$$

where $K(x)$ is the field obtained by adjoining all quotients x_i/x_j and $\Omega_{K(x)}$ is the set of places of $K(x)$. For details on why this is the right choice of height for such a point see Section 2.5, [83] and [1, §15.1]. The analogue of (4.1) was initially stated by Serre [74, Section 2.5] who gave a formula for the constant

$$S_K(n+1,1) = \left(\frac{1}{q^{g_K-1}}\right)^{n+1} \frac{J_K}{(q-1)\zeta_K(n+1)}, \tag{4.5}$$

where g_K is the genus of K, J_K is the number of divisor classes of degree 0 which is the cardinality of the Jacobian of K and ζ_K is the zeta function of K. Later Wan [86] and DiPippo [21] independently gave a proof of this result. Further work has been done by Thunder and Widmer [83] who obtained results for $d \geqslant 1$ and a more precise error term in the case $d = 1$. In particular, they prove that for $M \geqslant 2g_K - 1$ and $1/4 \geqslant \varepsilon > 0$, we have

$$N_K(3,1,M) = S_K(3,1)q^{3M} + O\left(q^{(M+g_K)(1+\varepsilon)}\right), \tag{4.6}$$

and if $M < 2g_K - 1$, then for all $\varepsilon > 0$, we have

$$N_K(3,1,M) \ll q^{M(2+\varepsilon)}, \tag{4.7}$$

where the implicit constants depends only on n, e, q and ε. In the case when $d = 2$, Kettlestrings and Thunder [46] improved the result of Thunder and Widmer [83] and showed that

$$N_K(3,2,M) = 2\left(S_K(3,1)\right)^2 q^{3M} M + O\left(q^{3M}\sqrt{M}\right) \tag{4.8}$$

where the implicit constant depends only on K.

4.2 GEOMETRY OF THE HILBERT SCHEME

Let K be a global field. Given a smooth projective variety V over K, we define the m^{th}*-symmetric product* of V to be the projective quotient variety

$$\operatorname{Sym}^m V = V^m/\mathfrak{S}_m,$$

where the symmetric group $\mathfrak{S}_m$ acts on V^m by permuting the factors. Denote by

$$\pi : V^m \to \operatorname{Sym}^m V$$

the canonical projection. Let $x \in V(\overline{K})$ be a point of degree $m \geqslant 1$, that is a point such that the degree of the residue field $\kappa(x)$ over K is equal to m. The orbit of x under the action of $\operatorname{Gal}(\overline{K}/K)$ contains exactly m distinct points $x_1, \ldots, x_m$ which are the conjugates of x. Then, as in [58], let $\tilde{x} = \pi(x_1, \ldots, x_m) \in \operatorname{Sym}^m V(\overline{K})$. Since $\tilde{x}$ is invariant under $\operatorname{Gal}(\overline{K}/K)$, it is rational over K. Le Rudulier [58, Definition 1.32] defines points in $\operatorname{Sym}^m V(K)$ of the shape $\tilde{x}$ to be *irreducible* and the rest to be *reducible*. Furthermore, according to [6, §6], if

$$\Delta = \bigcup_{1 \leqslant i < j \leqslant m} \{(v_1, \ldots, v_m) \in V^m \mid v_i = v_j\}$$

is the diagonal in V^m, then $\operatorname{Sym}^m V$ is singular along $D = \pi(\Delta)$.

Suppose from now on that V is a surface. In this case, by [6, §6], there is a resolution of singularities of $\operatorname{Sym}^m V$ given by the Hilbert-Chow birational morphism

$$\varepsilon : \operatorname{Hilb}^m V \to \operatorname{Sym}^m V$$

and $E = \varepsilon^{-1}(D)$ is an irreducible divisor of $\operatorname{Hilb}^m V$. Moreover, Fogarty [24, Theorem 2.4] proves that $\operatorname{Hilb}^m V$ is a smooth, irreducible projective variety of dimension $2m$ and ε induces an isomorphism

$$\operatorname{Hilb}^m V - E \cong \operatorname{Sym}^m V - D. \tag{4.9}$$

The Picard group of $\operatorname{Hilb}^m V$ is also computed by Fogarty [25, Theorem 6.2] (see [26] for characteristic 2) who proved that it has rank 2 and

$$\operatorname{Pic}\left(\operatorname{Hilb}^m \mathbb{P}^2\right) = \mathbb{Z}H \oplus \mathbb{Z}\frac{E}{2}, \tag{4.10}$$

where H is the locus of schemes intersecting a fixed line and E is exceptional divisor introduced above and corresponds to the locus of non-reduced schemes. This result allowed Huizenga [44, Theorem 1.4] to compute the cone of effective divisors for $\operatorname{Hilb}^m \mathbb{P}^2$. This is spanned by

$$\mu H - \frac{E}{2} \quad \text{and} \quad E, \tag{4.11}$$

where $\mu \in \mathbb{R}_{>0}$ is a certain minimum slope that will not be defined here. However, for $2 \leqslant m \leqslant 171$, the values of μ can be found in [44, Table 1], and by [44, Remark 7.4], and if m is of the form $\binom{r+2}{2}$, then $\mu = r$. We remark that in this notation, the anticanonical divisor of $\operatorname{Hilb}^m \mathbb{P}^2$ is $\omega^{-1}_{\operatorname{Hilb}^m \mathbb{P}^2} = 3H$.

Moreover, due to results of Beauville [6] in characteristic 0 and Kumar–Thomsen [53, Corollary 1] if $\operatorname{char}(K) > m$, ε is crepant. Thus, as noted in [58, Proposition 3.3 and §3.2], the anticanonical line bundle on $\operatorname{Hilb}^m V$ given by $\omega^{-1}_{\operatorname{Hilb}^m V} = \varepsilon^* \omega^{-1}_{\operatorname{Sym}^m V}$ is big since it is the pull-back of an ample divisor by a birational morphism. The anticanonical height on $\operatorname{Sym}^m V$ induces a height on $\operatorname{Hilb}^m V$ given by $H_{\omega^{-1}_{\operatorname{Hilb}^m V}} = H_{\omega^{-1}_{\operatorname{Sym}^m V}} \circ \varepsilon$. Now, as in [58, §1], given a point $v = \pi(v_1, \ldots, v_m) \in \operatorname{Sym}^m V(K)$, we define a height

$$H_{\omega^{-1}_{\operatorname{Sym}^m V}}(v) = H_{\omega_V^{-1}}(v_1) \ldots H_{\omega_V^{-1}}(v_m).$$

For the remainder of this section, let K be a function field of degree e over $\mathbb{F}_q(t)$ and H_n be the usual absolute height on projective space $\mathbb{P}^n$ with respect to the ground field

$\mathbb{F}_q(t)$ as in (4.4). Regarding $\mathbb{P}^n$ over the ground field K, we have that for $x \in \mathbb{P}^n(\overline{K})$, the absolute height on projective space associated to the anticanonical line bundle is

$$H_{\omega_{\mathbb{P}^n}^{-1}}(x) = H_n(x)^{(n+1)e}.$$

We are mainly interested in the case when $m = 2$, $V = \mathbb{P}^2$ is defined over K and $\mathrm{char}(K) > 2$. In order to study K-points on $\mathrm{Hilb}^2 \mathbb{P}^2$ it is convenient to use the height function

$$H(z) = H_{\omega_{\mathrm{Hilb}^2 \mathbb{P}^2}^{-1}}(z)^{1/3}. \tag{4.12}$$

Theorem 4.2.1. *Let K be a function field of degree e over $\mathbb{F}_q(t)$ of characteristic > 2. Let $Z_0 = \varepsilon^{-1}\pi(\mathbb{P}^2 \times \mathbb{P}^2(K))$. Suppose $M \geqslant 2(2g_K - 1)$. Then, for all non-empty open subsets U of $\mathrm{Hilb}^2 \mathbb{P}^2$ we have*

$$\#\left\{z \in (Z_0 \backslash E(K)) \cap U(K) : H(z) = q^M\right\} = \frac{S_K(3,1)^2}{2} q^{3M} M + O\left(q^{3M}\right),$$

where $S_K(3,1)$ is given by (4.5), and

$$\#\left\{z \in \mathrm{Hilb}^2 \mathbb{P}^2(K) \backslash Z_0 : H(z) = q^M\right\} = S_K(3,1)^2 q^{3M} M + O\left(\sqrt{M} q^{3M}\right),$$

as $M \to \infty$. In particular, in the second part, the leading constant agrees with the prediction of Peyre and the implicit constant in the error term depends only on K.

Thus, after removing the thin set Z_0, Manin's conjecture holds for $\mathrm{Hilb}^2 \mathbb{P}^2$. The proof can be found in Section 4.4. The first part relies on (4.6) and (4.7), which is where the condition that $M \geqslant 2(2g_K - 1)$ comes from, and the second part uses (4.8). Theorem 4.1.2 follows from taking (4.12) in Theorem 4.2.1.

This can be seen as a more general function field analogue of Le Rudulier's result [58, Theorem 4.2] which states that there exists a thin set $Z_0 \subset \mathrm{Hilb}^2 \mathbb{P}^2(\mathbb{Q})$ such that for all non-empty open subsets U of $\mathrm{Hilb}^2 \mathbb{P}^2$ we have

$$\#\left\{z \in (Z_0 \backslash E(\mathbb{Q})) \cap U(\mathbb{Q}) : H_{\omega_{\mathrm{Hilb}^2 \mathbb{P}^2, \mathbb{Q}}^{-1}}(z) \leqslant B\right\} \sim \frac{8}{\zeta(3)^2} B \log B,$$

and

$$\#\left\{z \in \mathrm{Hilb}^2 \mathbb{P}^2(\mathbb{Q}) \backslash Z_0 : H_{\omega_{\mathrm{Hilb}^2 \mathbb{P}^2, \mathbb{Q}}^{-1}}(z) \leqslant B\right\} \sim \frac{24 + 2\pi^2}{3\zeta(3)^2} B \log B,$$

as $B \to \infty$.

4.3 MANIN'S CONJECTURE AND PEYRE'S CONSTANT OVER FUNCTION FIELDS

Given a Fano variety V defined over a function field K, let

$$N_V(M) := \#\left\{P \in V(K) : \mathcal{H}(P) = q^M\right\},$$

where $\mathcal{H}$ denotes the anticanonical height $H_{\omega_V^{-1}}$. Define the anticanonical height zeta function of V to be

$$Z_{\mathcal{H}}(s) = \sum_{y \in V(K)} \frac{1}{\mathcal{H}(y)^s}.$$

Then, by an application of the Wiener–Ikehara theorem [70, Theorem 17.4] we obtain what is known as Manin's conjecture, that is

$$N_V(M) \sim c_{\mathcal{H}}(V) \frac{(\log q)^{r_V}}{(r_V - 1)!} q^M M^{r_V - 1}, \tag{4.13}$$

as $M \to \infty$, where

$$c_{\mathcal{H}}(V) = \lim_{s \to 1} (s-1)^{r_V} Z_{\mathcal{H}}(s). \tag{4.14}$$

and the rank r_V of the Picard group of V is equal to the the multiplicity of the pole of $Z_{\mathcal{H}}(s)$ of V at $s = 1$. For more details on Manin's conjecture and Peyre's refinements, we refer the reader to Section 2.4. We will only remind here that the leading constant above has been given a geometric interpretation by Peyre (see [65, 9, 10, 67]). In particular, the constant is given by

$$c_{\mathcal{H}}(V) = \alpha^*(V)\beta(V)\tau_{\mathcal{H}}(V). \tag{4.15}$$

Lemma 4.3.1. *Let $m \in \mathbb{Z}_{\geqslant 2}$ and K be a degree $e \geqslant 1$ extension of $\mathbb{F}_q(t)$ such that $\operatorname{char}(K) > m$. Then, Peyre's constant for $\operatorname{Hilb}^m \mathbb{P}^2$ defined over K is given by*

$$c_{\mathcal{H}}(\operatorname{Hilb}^m \mathbb{P}^2) = \frac{\mu J_K^2}{9(q-1)^2 q^{2(m+1)(g_K-1)} (\log q)^2} \prod_{v \in \Omega_K} \left(1 - q_v^{-1}\right)^2 \frac{\left|\operatorname{Hilb}^m \mathbb{P}^2(\mathbb{F}_{q_v})\right|}{q^{2m}},$$

where g_K is the genus of K, J_K is the number of divisor classes of degree 0 which is the cardinality of the Jacobian of K, ζ_K is the zeta function of K and μ is as in (4.11).

Proof. By (2.11), (4.10) and (4.11), and since the anticanonical divisor is $\omega^{-1}_{\operatorname{Hilb}^m \mathbb{P}^2} = 3H$, we have that

$$C^{\vee}_{\text{eff}}(\operatorname{Hilb}^m \mathbb{P}^2) = \left\{aH + \frac{b}{\mu}E, \left(a, \frac{b}{\mu}\right) \in \mathbb{R}^2 : a - \frac{b}{\mu} \geqslant 0, \frac{b}{\mu} \geqslant 0\right\},$$

Then, by (2.10), and since $\mu \in \mathbb{R}_{>0}$, we obtain

$$\alpha^*(\operatorname{Hilb}^m \mathbb{P}^2) = \int_{C_{\text{eff}}^\vee(\operatorname{Hilb}^m \mathbb{P}^2)} e^{-\langle 3H, y\rangle}\, \mathrm{d}y = \int_0^\infty \mathrm{d}b \int_{\frac{b}{\mu}}^\infty e^{-3a}\, \mathrm{d}a = \frac{\mu}{3^2}.$$

By (2.12), we have $\beta(\operatorname{Hilb}^m \mathbb{P}^2) = 1$. Given a place $v \in \Omega_K$, let κ_v be the residue field with respect to v and $\#\kappa_v = q_v$. Since $\operatorname{rk}\operatorname{Pic}\operatorname{Hilb}^m \mathbb{P}^2 = 2$ and $\dim \operatorname{Hilb}^m \mathbb{P}^2 = 2m$, and by (2.13), then

$$\tau_{\mathcal{H}}(\operatorname{Hilb}^m \mathbb{P}^2) = q^{2m(1-g_K)} \lim_{s\to 1}(s-1)^2 \zeta_K(s)^2 \prod_{v\in\Omega_K} \left(1-q_v^{-1}\right)^2 \omega_v(\operatorname{Hilb}^m \mathbb{P}^2),$$

where

$$\omega_v(\operatorname{Hilb}^m \mathbb{P}^2) = \frac{|\operatorname{Hilb}^m \mathbb{P}^2(\mathbb{F}_{q_v})|}{{q_v}^{2m}}.$$

As in (2.14) in Example 2.4.4, we have that

$$\lim_{s\to 1}(s-1)^2\zeta_K(s)^2 = \left(\lim_{s\to 1} L_K(q^{-s})\zeta_{\mathbb{F}_q(t)}(s)\right)^2 = \left(\frac{J_K q^{1-g_K}}{(q-1)\log q}\right)^2.$$

Thus, to compute $\tau_{\mathcal{H}}(\operatorname{Hilb}^m \mathbb{P}^2)$, it is enough to know $|\operatorname{Hilb}^m \mathbb{P}^2(\mathbb{F}_q)|$. A formula for this, involving Betti numbers, has first been given by Ellingsrud–Strømme [22, Theorem 1.1]. However, it can be computed using a more general result of Göttsche [31, Lemma 2.3.9] which implies that

$$\sum_{m=0}^\infty \left|\operatorname{Hilb}^m \mathbb{P}^2(\mathbb{F}_q)\right| t^m = \exp\left(\sum_{k=1}^\infty \frac{t^k}{k} \frac{\left|\mathbb{P}^2(\mathbb{F}_{q^k})\right|}{1-q^k t^k}\right).$$

By [22, Corollary 1.3], we have that for $m \geqslant 2$, $\left|\operatorname{Hilb}^m \mathbb{P}^2(\mathbb{F}_q)\right| = q^{2m} + 2q^{2m-1} + O\left(q^{2m-2}\right)$. It follows that $\omega_v(\operatorname{Hilb}^m \mathbb{P}^2) = 1 + 2q_v^{-1} + O\left(q_v^{-2}\right)$, which is what we expect by Manin's conjecture; hence, the product $\prod_v \lambda_v^{-1}\omega_v$, where $\lambda_v^{-1} = (1-q_v^{-1})^2$, converges. We remark that, by the isomorphism (4.9), we have

$$\left|\operatorname{Hilb}^m \mathbb{P}^2(\mathbb{F}_q)\right| = \left|\operatorname{Sym}^m \mathbb{P}^2(\mathbb{F}_q)\right| - |D(\mathbb{F}_q)| + |E(\mathbb{F}_q)|.$$

Hence, we could also compute $\left|\operatorname{Hilb}^m \mathbb{P}^2(\mathbb{F}_q)\right|$ by using Lemma 5.2.1, if we understood the geometry of the exceptional divisor E, which has been studied independently by Iarrobino [45] and Briançon [11]. One can find an explicit description for the cases when $1 \leqslant m \leqslant 6$ in [11, §IV.2]. However, for $m > 2$, we could not find a nice interpretation for $\prod_{v\in\Omega_K}\left(1-q_v^{-1}\right)^2 \omega_v(\operatorname{Hilb}^m \mathbb{P}^2)$. $\square$

Corollary 4.3.2. *Let K be a degree $e \geqslant 1$ extension of $\mathbb{F}_q(t)$ such that $\mathrm{char}(K) > 2$. Then Peyre's constant for $\mathrm{Hilb}^2\,\mathbb{P}^2$ defined over K is given by*

$$c_{\mathcal{H}}(\mathrm{Hilb}^2\,\mathbb{P}^2) = \frac{S_K(3,1)^2}{9\,(\log q)^2},$$

where $S_K(3,1)$ is given by (4.5).

Proof. This follows by taking $m = 2$ in Lemma 4.3.1. By [44, Table 1], we have that $\mu = 1$. Moreover, by [22, Table 1], we obtain that

$$\omega_v(\mathrm{Hilb}^2\,\mathbb{P}^2) = \frac{\#\,\mathrm{Hilb}^2\,\mathbb{P}^2(\mathbb{F}_{q_v})}{q_v^4} = 1 + 2q_v^{-1} + 3q_v^{-2} + 2q_v^{-3} + q_v^{-4},$$

for all $v \in \Omega_K$. Thus, taking $\lambda_v^{-1} = (1-q_v^{-1})^2$, we obtain $\prod_v \lambda_v^{-1}\omega_v = \zeta_K(3)^{-2}$. Hence,

$$c_{\mathcal{H}}(\mathrm{Hilb}^2\,\mathbb{P}^2) = \frac{J_K^2}{9(q-1)^2 q^{6(g_K-1)}(\log q)^2 \zeta_K(3)^2},$$

and the result follows by (4.5). □

4.4 PROOF OF THE MAIN THEOREM

In this section, we prove Theorem 4.2.1. Recall that $[K : \mathbb{F}_q(t)] = e$, $\mathrm{char}(K) > 2$ and $Z_0 = \varepsilon^{-1}\pi(\mathbb{P}^2 \times \mathbb{P}^2(K))$. For the proof of the first part, define the height function

$$H_{\mathbb{P}^2\times\mathbb{P}^2} : (x_1, x_2) \to H_2(x_1)^e H_2(x_2)^e.$$

Then, as in [58, Proposition 3.14], $\#\left\{z \in (Z_0 \backslash E(K)) \cap U(K) : H(z) = q^M\right\}$ is equal to

$$\begin{aligned}
&\#\left\{z \in \varepsilon^{-1}\pi\left(\mathbb{P}^2 \times \mathbb{P}^2\right)(K) \cap (\mathrm{Hilb}^2\,\mathbb{P}^2 - E)(K) \cap U(K) : H(z) = q^M\right\}\\
=&\#\left\{z \in V_r \cap \varepsilon((\mathrm{Hilb}^2\,\mathbb{P}^2 - E) \cap U)(K) : H(z) = q^M\right\}\\
=&\frac{1}{2}\#\left\{(x,y) \in \left(\mathbb{P}^2 \times \mathbb{P}^2\right)(K) \cap \pi^{-1}\varepsilon((\mathrm{Hilb}^2\,\mathbb{P}^2 - E) \cap U)(K) : H_{\mathbb{P}^2\times\mathbb{P}^2}(x,y) = q^M\right\},
\end{aligned}$$

where $V_r = \pi\left(\mathbb{P}^2 \times \mathbb{P}^2(K)\right)$ is the set of reducible points of $\mathrm{Sym}^2\,\mathbb{P}^2(K)$. Rewrite $\frac{1}{2}\#\left\{(x,y) \in \left(\mathbb{P}^2 \times \mathbb{P}^2\right)(K) : H_{\mathbb{P}^2\times\mathbb{P}^2}(x,y) = q^M\right\}$ as

$$\frac{1}{2}\sum_{N=0}^{M} \#\left\{x \in \mathbb{P}^2(K) : H_2(x) = q^{\frac{N}{e}}\right\} \#\left\{y \in \mathbb{P}^2(K) : H_2(y) = q^{\frac{M-N}{e}}\right\}. \tag{4.16}$$

If $e = 1$, then $K = \mathbb{F}_q(t)$ and

$$S_{\mathbb{F}_q(t)}(3,1) = \frac{(q^3-1)(1-q^{-2})}{q-1},$$

by (4.5). Then, by (2.24), we have that (4.16) is equal to

$$\begin{aligned}
&\frac{1}{2}\left(\frac{2(q^3-1)}{q-1}S_{\mathbb{F}_q(t)}(3,1)q^{3M}+\sum_{N=1}^{M-1}S_{\mathbb{F}_q(t)}(3,1)^2q^{3M}\right)\\
=&\frac{1}{2}\left(\frac{2}{1-q^{-2}}S_{\mathbb{F}_q(t)}(3,1)^2q^{3M}+(M-1)S_{\mathbb{F}_q(t)}(3,1)^2q^{3M}\right)\\
=&\frac{S_{\mathbb{F}_q(t)}(3,1)^2}{2}q^{3M}M+\frac{q^2+1}{2(q^2-1)}S_{\mathbb{F}_q(t)}(3,1)^2q^{3M}.
\end{aligned}$$

Otherwise, if $e \geqslant 1$, we split the sum over N into three sums: the first sum runs over $0 \leqslant N \leqslant 2g_K - 2$, the second over $2g_K - 1 \leqslant N \leqslant M - 2g_K + 1$, and the last over the remaining $N \leqslant M$. Thus, by the result of Thunder and Widmer (4.6) and (4.7), the first and third sum are each $\ll S_K(3,1)q^{3M}$, where the implicit constant depends on e and q. For the second sum we note that both N and $M - N$ are $\geqslant 2g_K - 1$ and hence we obtain it is equal to

$$\left(\frac{M+1}{2}-(2g_K-1)\right)S_K(3,1)^2q^{3M}+O\left(S_K(3,1)q^{3M-3g_K+2}\right).$$

Thus, in both cases above we obtain

$$\frac{1}{2}\#\left\{(x,y)\in\left(\mathbb{P}^2\times\mathbb{P}^2\right)(K): H(x,y)=q^M\right\}=\frac{S_K(3,1)^2}{2}q^{3M}M+O\left(q^{3M}\right).$$

Moreover, the contribution to $\frac{1}{2}\#\left\{(x,y)\in\left(\mathbb{P}^2\times\mathbb{P}^2\right)(K): H_{\mathbb{P}^2\times\mathbb{P}^2}(x,y)=q^M\right\}$ coming from a proper closed subset of $\mathbb{P}^2\times\mathbb{P}^2$ is at most

$$=\frac{1}{2}\sum_{N=0}^{M}\#\left\{x\in\mathbb{P}^1(K): H_1(x)=q^{\frac{N}{e}}\right\}\#\left\{y\in\mathbb{P}^2(K): H_2(y)=q^{\frac{M-N}{e}}\right\}. \tag{4.17}$$

Following the same argument as above, if $K = \mathbb{F}_q(t)$, (4.17) is equal to

$$\frac{q^2-1}{2(q-1)}S_{\mathbb{F}_q(t)}(3,1)q^{3M}+\frac{1}{2}S_{\mathbb{F}_q(t)}(2,1)S_{\mathbb{F}_q(t)}(3,1)q^{3M}\sum_{N=1}^{M-1}q^{-N}+\frac{q^3-1}{2(q-1)}S_{\mathbb{F}_q(t)}(2,1)q^{2M}.$$

If $[K : \mathbb{F}_q(t)] > 1$, we split the sum over N into three sums as before which contribute $\ll S_K(3,1)q^{3M}$, $\sim S_K(3,1)S_K(2,1)q^{3M-2g_K+1}$, and $\ll S_K(2,1)q^{2M}$, respectively. Hence, the contribution from proper closed subsets of $\mathbb{P}^2\times\mathbb{P}^2$ is $O(q^{3M})$ and this concludes the proof of the first part of the theorem.

The proof of the second part is similar to the argument over number fields. Using the isomorphism

$$\operatorname{Hilb}^2\mathbb{P}^2\backslash E \xrightarrow{\sim} \operatorname{Sym}^2\mathbb{P}^2\backslash D,$$

we note that it suffices to study rational points of $\mathrm{Sym}^2 \mathbb{P}^2 \backslash D$ of bounded height. Let $\overline{x} \in \mathbb{P}^2(\overline{K})$ be the conjugate of the quadratic point x. Then we have that

$$\# \left\{ y \in \left(\mathrm{Sym}^2 \mathbb{P}^2 \backslash V_r \right) (K) : H(y) = q^M \right\}$$

is equal to

$$\frac{1}{2} \# \left\{ (x, \overline{x}) \in \left(\mathbb{P}^2 \times \mathbb{P}^2 \right) (\overline{K}) : [K(x) : K] = 2, H_2(x)^e H_2(\overline{x})^e = q^M \right\}.$$

Since by [58, Proposition 1.17], the height is invariant under Galois conjugation, this reduces to counting quadratic points in $\mathbb{P}^2$. Thus, it is equal to

$$\begin{aligned} &\frac{1}{2} \# \left\{ x \in \mathbb{P}^2(\overline{\mathbb{F}_q(t)}) : [K(x) : K] = 2, H_2(x) = q^{\frac{M}{2e}} \right\} \\ =& S_K(3,1)^2 q^{3M} M + O\left(\sqrt{M} q^{3M} \right), \end{aligned}$$

by the result of Kettlestring and Thunder (4.8).

Now we verify that the leading constant we obtain in the second part agrees with the prediction of Peyre. Over function fields, Manin's conjecture predicts that for a Fano variety X with associated anticanonical height $\mathcal{H}$, we have that $\lim_{s \to 1} (s-1)^r \mathcal{Z}_X(s)$, where $\mathcal{Z}_X(s)$ is the anticanonical height zeta function associated to X and $r = \mathrm{rk}\,\mathrm{Pic}\, X$, is equal to Peyre's constant $c_{\mathcal{H}}(X)$ as given by (4.15) and that the multiplicity of the pole of $\mathcal{Z}_X$ at $s = 1$ equals r. In our case $X(K) = \mathrm{Hilb}^2 \mathbb{P}^2(K)^{\mathrm{irr}}$, since we only consider the irreducible points, and $\mathrm{rk}\,\mathrm{Pic}\,\mathrm{Hilb}^2 \mathbb{P}^2 = 2$. The height zeta function corresponding to the height H defined by (4.12) satisfies

$$Z_X^{\mathrm{irr}}(3s) = \mathcal{Z}_X^{\mathrm{irr}}(s) \tag{4.18}$$

and thus, Z_X^{irr} has a double pole at $s = 1$. Hence, we consider $Z_X^{\mathrm{irr}}(s+2)$ which, by Wiener–Ikehara theorem [70, Theorem 17.4], leads us to expect

$$q^{-2M} \# \left\{ z \in X(K) : H(z) = q^M \right\} \sim (\log q)^2 q^M M \lim_{s \to 1} (s-1)^2 Z_X^{\mathrm{irr}}(s+2),$$

as $M \to \infty$. Moreover, Lemma 4.3.2 together with (4.18) implies that $c_H(X) = 9 c_{\mathcal{H}}(X)$ and thus, the conjecture predicts that the number of K-points z on $\mathrm{Hilb}^2 \mathbb{P}^2$ of height $H(z) = q^M$ is

$$\sim \frac{S_K(3,1)^2}{(\log q)^2} \cdot (\log q)^2 q^{3M} M,$$

as $M \to \infty$, which concludes the proof.

5

THE PRIME NUMBER THEOREM FOR 0-CYCLES

5.1 INTRODUCTION

The analogy between integers and polynomials in one variable with coefficients in a finite field $\mathbb{F}_q$ presented in Section 2.1 can be extended to an analogy between positive integers and effective 0-cycles on a variety V over $\mathbb{F}_q$, i.e. formal sums $C = \sum_{i=1}^{k} n_i P_i$, where $n_i \in \mathbb{N}$ and P_i are distinct points on V. In particular, primes correspond to points P_i. This is a natural generalisation, since effective 0-cycles on the affine line correspond to polynomials and it would be interesting to understand the relation with 0-cycles on other varieties. We recall that if K is a field, $x = [x_0 : \ldots : x_n]$ is a point in $\mathbb{P}^n(K)$, then $K(x)$ is the field obtained by adjoining all quotients x_i/x_j and the degree of x is given by $[K(x) : K]$. We define the degree of a 0-cycle $C = \sum_{i=1}^{k} n_i P_i$ is $\sum_{i=1}^{k} n_i \deg P_i$. In view of this analogy, Chen [19] establishes several results for 0-cycles that can be seen as an equivalent to classical theorems in analytic number theory. For example, the classical prime number theorem states that the probability that an integer between e^m and $2e^m$ chosen uniformly at random to be prime is $\sim 1/m$ as $m \to \infty$. The analogue for 0-cycles is given by [19, Theorem 1] and it says that for a geometrically connected variety V over $\mathbb{F}_q$ of dimension $d \geqslant 1$, we have

$$\frac{\#\{\text{points on } V \text{ of degree } m\}}{\#\{\text{effective } 0\text{-cycles on } V \text{ of degree } m\}} = \frac{1}{m\mathring{Z}(V, q^{-d})} + O\left(\frac{1}{mq^{m/2}}\right), \tag{5.1}$$

as $m \to \infty$, where $\mathring{Z}(V,t) := Z(V,t)(1 - q^d t)$ and $Z(V,t)$ is the zeta function of V over $\mathbb{F}_q$.

An effective 0-cycle of degree m on V can also be thought of as a point in $\operatorname{Sym}^m V(\mathbb{F}_q)$ in the following way. As explained in [19], a point in $\operatorname{Sym}^m V(\mathbb{F}_q)$ can be viewed as a multiset $\{x_1, \ldots, x_n\}$ of points in $V(\mathbb{F}_q)$, preserved by the action of Frobenius and where repetition is allowed. Thus, we can decompose this multiset into a union of orbits of Frobenius, say n_1 orbits of size $k_1, \ldots n_j$ orbits of size k_j such that $n_1 k_1 + \ldots + n_j k_j = n$. However, orbits of size k are in bijection with degree k points on V. Hence, such a multiset corresponds to a 0-cycle $C = \sum_{i=1}^{k} n_i P_i$ on V where $\deg P_i = k_i$ for all i. Thus, using a result of Göttsche [31, Remark 1.2.4], we improve the error term in (5.1) for the particular case when $V = \mathbb{P}^2$.

Theorem 5.1.1. *The proportion of prime effective 0-cycles of degree m on $\mathbb{P}^2$ to all effective 0-cycles of degree m on $\mathbb{P}^2$ is*

$$\frac{1}{m}\left(1 - q^{-1} - q^{-2} + q^{-3}\right) + O\left(\frac{1}{mq^m}\right),$$

as $m \to \infty$.

Inspired by this, we consider a similar problem for K-points of $\operatorname{Sym}^m \mathbb{P}^2$, where K is a finite extension of $\mathbb{F}_q(t)$, and thus, obtain a version of the prime number theorem for 0-cycles on $\mathbb{P}^2$ over function fields. The prime 0-cycles of degree m on $\mathbb{P}^2$ are precisely the K-points of degree m of $\mathbb{P}^2$. As described in Section 4.2, if $x \in \mathbb{P}^2(\overline{K})$ is of degree $m \geqslant 1$, then it has m distinct Galois conjugates $x_1, \ldots, x_m$ and $\tilde{x} = \pi(x_1, \ldots, x_m)$ is a K-point of $\operatorname{Sym}^m \mathbb{P}^2$. Thus, primes correspond to the irreducible K-points of $\operatorname{Sym}^m \mathbb{P}^2$. We also notice that in this case, one needs to consider points of bounded height since the set of points $\operatorname{Sym}^m \mathbb{P}^2(K)$ is infinite.

Corollary 5.1.2. *Let $m \geqslant 2$ and K be a global function field of characteristic $> m$. Suppose that Manin's conjecture holds for the irreducible points in $\operatorname{Hilb}^{m_0} \mathbb{P}^2(K)$ for all $m_0 \leqslant m$. Then, there exists a constant $c_m > 0$ such that the proportion of effective 0-cycles on $\mathbb{P}^2$ over K of degree m corresponding to K-points on $\operatorname{Sym}^m(\mathbb{P}^2)$ of height q^M that are prime is*

$$\sim \frac{c_m}{M^{m-2}},$$

$M \to \infty$, where $c_2 = \frac{2}{3}$ and

$$c_m = \frac{\mu 3^{m-2} \zeta_K(3)^2 m!(m-1)!}{S_K(3,1)^{m-2}} \prod_{v \in \Omega_K} \left(1 - q_v^{-1}\right)^2 \frac{\left|\operatorname{Hilb}^m \mathbb{P}^2(\mathbb{F}_{q_v})\right|}{q^{2m}},$$

if $m \geqslant 3$.

When $m = 2$, the result is actually unconditional due to Theorem 4.2.1.

5.2 0-CYCLES ON $\mathbb{P}^2$ OVER $\mathbb{F}_q$.

In this section, we present an improvements to [19, Theorem 1] for the special case when $V = \mathbb{P}^2$. We begin with some technical lemmas.

Lemma 5.2.1. *The number of effective* 0*-cycles of degree* m *on* $\mathbb{P}^2$ *over* $\mathbb{F}_q$ *is*

$$\left(1+\frac{1}{q}\right)\sum_{i=0}^{k-1}(i+1)(q^{2(m-i)}+q^{2i+1})+\begin{cases}\frac{m+2}{2}q^m, & \text{if } m=2k,\\ \frac{m+1}{2}q^{m-1}\left(q^2+q+1\right), & \text{if } m=2k+1.\end{cases}$$

Proof. Effective 0-cycles of degree m on $\mathbb{P}^2$ over $\mathbb{F}_q$ correspond to $\mathbb{F}_q$-points on $\mathrm{Sym}^m\,\mathbb{P}^2$. By [31, Remark 1.2.4], we have

$$\sum_{m=0}^{\infty}\left|\mathrm{Sym}^m\,\mathbb{P}^2(\mathbb{F}_q)\right|t^m=\exp\left(\sum_{k=1}^{\infty}\left|\mathbb{P}^2(\mathbb{F}_{q^k})\right|\frac{t^k}{k}\right)=Z(\mathbb{P}^2,t),$$

the zeta function of $\mathbb{P}^2$ over $\mathbb{F}_q$. This is $(1-q^2t)^{-1}(1-qt)^{-1}(1-t)^{-1}$. Thus, by using the Taylor expansion at $t=0$ we obtain the claimed result. □

Lemma 5.2.2. *The number of prime effective* 0*-cycles of degree* m *on* $\mathbb{P}^2$ *over* $\mathbb{F}_q$ *is*

$$\begin{cases}\frac{1}{m}\left(q^{2m}-q^{\frac{m}{2}}\right), & \text{if } m \text{ is even},\\ \frac{1}{m}\left(q^{2m}+q^m-q^{\frac{2m}{j}}-q^{\frac{m}{j}}\right), & \text{if } m \text{ is odd},\end{cases}$$

where j *is the smallest divisor of* m *that is greater than 1.*

Proof. By [31, Definition 1.2.3, Remark 1.2.4(1)], the number of prime effective 0-cycles of degree m on $\mathbb{P}^2$ over $\mathbb{F}_q$ is

$$\frac{q^{2m}+q^m+1}{m}-\frac{1}{m}\sum_{\substack{r\mid m\\ r\neq m}}r\left|P_r(\mathbb{P}^2,\mathbb{F}_q)\right|, \tag{5.2}$$

where $P_r(\mathbb{P}^2,\mathbb{F}_q)$ are the prime effective 0-cycles of degree r on $\mathbb{P}^2$ over $\mathbb{F}_q$. If m is even, write $m=2k$. Then, k is the largest proper divisor of m, and (5.2) becomes

$$\frac{q^{2m}+q^m+1}{m}-\frac{1}{m}\left(q^{2k}+q^k+1-\sum_{\substack{s\mid k\\ s\neq k}}s\left|P_s(\mathbb{P}^2,\mathbb{F}_q)\right|\right)-\frac{1}{m}\sum_{\substack{r\mid m\\ r\neq k,m}}r\left|P_r(\mathbb{P}^2,\mathbb{F}_q)\right|,$$

and hence the result. If m is odd, the proof is similar. □

Proof of Theorem 5.1.1. Theorem 5.1.1 follows immediately from the previous lemmas. More precisely, in the case when m is even, write $m=2k$, for $k\in\mathbb{Z}_{>0}$. Then, we compute

$$\begin{aligned}&(1-q^{-1}-q^{-2}+q^{-3})\left(\left(1+\frac{1}{q}\right)\sum_{i=0}^{k-1}(i+1)(q^{2(m-i)}+q^{2i+1})+\frac{m+2}{2}q^m\right)\\ =&q^{2m}-q^{m-1}+O\left(q^{m-2}\right),\end{aligned}$$

and, by Lemma 5.2.2, we obtain that the proportion of prime effective 0-cycles of degree m on $\mathbb{P}^2$ is $\frac{1}{m}\left(1-q^{-1}-q^{-2}+q^{-3}\right)+O\left(\frac{1}{mq^{m+1}}\right)$. If m is odd, then writing $m=2k+1$, where $k\in\mathbb{Z}\geqslant 0$, we have

$$\left(1-q^{-1}-q^{-2}+q^{-3}\right)\left(\left(1+\frac{1}{q}\right)\sum_{i=0}^{k-1}(i+1)(q^{2(m-i)}+q^{2i+1})+\frac{m+1}{2}q^{m-1}\left(q^2+q+1\right)\right)$$
$$=q^{2m}-q^{m-1}+O\left(q^{m-2}\right),$$

and, by Lemma 5.2.2, we obtain that the proportion of primes is $\frac{1}{m}\left(1-q^{-1}-q^{-2}+q^{-3}\right)+O\left(\frac{1}{mq^{m}}\right)$, as $m\to\infty$. $\square$

Comparing this with (5.1) which states that the proportion of prime effective 0-cycles of degree m on $\mathbb{P}^2$ is $\frac{1}{m}\left(1-q^{-1}-q^{-2}+q^{-3}\right)+O\left(\frac{1}{mq^{m/2}}\right)$, as $m\to\infty$, we notice that in this particular case we manage to obtain a better error term.

5.3 0-cycles on $\mathbb{P}^2$ over function fields.

In this section, we present an application of the main result in Chapter 4 which can be seen as an analogue of the prime number theorem for 0-cycles on $\mathbb{P}^2$ over function fields. By Theorem 4.2.1, if K is a function field of characteristic >2, then Manin's conjecture holds for the irreducible points of $\operatorname{Hilb}^2\mathbb{P}^2(K)$, which were introduced in Section 4.2. Throughout this section, we will assume that Manin's conjecture holds for the irreducible points in $\operatorname{Hilb}^m\mathbb{P}^2(K)$ for all $m\geqslant 3$, where K is a function field of characteristic $>m$. Let

$$N_{\operatorname{Sym}^m\mathbb{P}^{2\,\mathrm{irr}}}(M)=\left\{x\in\operatorname{Sym}^m\mathbb{P}^2(K): x \text{ irreducible}, H_{\omega^{-1}_{\operatorname{Sym}^m\mathbb{P}^2}}(x)=q^M\right\}.$$

By the isomorphism given in (4.9) and Theorem 4.2.1, we have

$$N_{\operatorname{Sym}^m\mathbb{P}^{2\,\mathrm{irr}}}(M)\sim\begin{cases}\frac{1}{9}S_K(3,1)^2q^MM, & \text{if } m=2,\\ c_{\omega^{-1}_{\operatorname{Hilb}^m\mathbb{P}^2}}(\operatorname{Hilb}^m\mathbb{P}^2)\left(\log q\right)^2q^MM, & \text{if } m>2,\end{cases}\tag{5.3}$$

as $M\to\infty$, since $\operatorname{rk}\operatorname{Pic}\operatorname{Hilb}^m\mathbb{P}^2=\operatorname{rk}\operatorname{Pic}\mathbb{P}^2+1=2$ for all $m\geqslant 2$. By understanding the various types of non-irreducible points of $\operatorname{Sym}^m\mathbb{P}^2$, we shall obtain the following main result, which together with (5.3) and Lemma 4.3.1, implies Corollary 5.1.2.

Theorem 5.3.1. *Let m be an integer $\geqslant 2$ and K be a global function field of characteristic $> m$. Suppose that Manin's conjecture holds for the irreducible points in $\mathrm{Hilb}^{m_0}\,\mathbb{P}^2(K)$ for all $3 \leqslant m_0 \leqslant m$. Then,*

$$N_{\mathrm{Sym}^m \mathbb{P}^2}(M) \sim \frac{S_K(3,1)^2}{6} q^M M, \text{ if } m = 2,$$

and

$$N_{\mathrm{Sym}^m \mathbb{P}^2}(M) = \frac{S_K(3,1)^m}{3^m m!(m-1)!} q^M M^{m-1} + O(q^M M^{m-3}), \text{ if } m > 2.$$

as $M \to \infty$.

To prove Theorem 5.3.1, we will require the following results. Similar to the case of number fields, Manin's conjecture and Peyre's prediction are compatible with products of varieties over function fields.

Theorem 5.3.2. *Let V, W be two Fano varieties defined over a function field K such that $V \times W(K) \neq \varnothing$ and $r_V \geqslant r_W$. Assume that (4.13) and (4.14) hold for V and W. Then, we have that $N_{V\times W}(M)$ is*

$$\sim c_{H_{\omega^{-1}_{V\times W}}}(V \times W) \frac{(\log q)^{r_{V\times W}}}{(r_{V\times W} - 1)!} q^M M^{r_{V\times W}-1},$$

as $M \to \infty$, where

$$c_{H_{\omega^{-1}_{V\times W}}}(V \times W) = \lim_{s\to 1}(s-1)^{r_{V\times W}} Z_{H_{\omega^{-1}_{V\times W}}}(s)$$

agrees with the constant predicted by Peyre and the rank $r_{V\times W}$ of the Picard group of $V \times W$ is equal to the the multiplicity of the pole of the anticanonical height zeta function $Z_{H_{\omega^{-1}_{V\times W}}}(s)$ at $s = 1$.

Proof. As in the proof of [65, Lemma 3.0.2], we have an isomorphism

$$\mathrm{Pic}\, V \times \mathrm{Pic}\, W \to \mathrm{Pic}(V \times W). \tag{5.4}$$

Thus, $r_{V\times W} = r_V + r_W$ and the metric on the anticanonical divisor $\omega^{-1}_{V\times W}$ is the product of the metrics on ω_V^{-1} and ω_W^{-1}. Hence, the height of a point $(x, y) \in V \times W(K)$ is given by $H_{\omega^{-1}_{V\times W}}(x, y) = H_{\omega_V^{-1}}(x) H_{\omega_W^{-1}}(y)$. This implies that we can write $N_{V\times W}(M)$ as

$$\begin{aligned}&\sum_{i=0}^{M} \#\left\{x \in V(K) : H_{\omega_V^{-1}}(x) = q^i\right\} \#\left\{y \in W(K) : H_{\omega_W^{-1}}(y) = q^{M-i}\right\} \\ \sim &\frac{c_{H_{\omega_V^{-1}}}(V) c_{H_{\omega_W^{-1}}}(W)}{(r_V - 1)!(r_W - 1)!} (\log q)^{r_V + r_W} q^M \sum_{i=0}^{M} i^{r_V - 1}(M - i)^{r_W - 1},\end{aligned} \tag{5.5}$$

as $M \to \infty$. Replacing the sum over i by an integral, we obtain that the above is

$$\sim \frac{c_{H_{\omega_V^{-1}}}(V)c_{H_{\omega_W^{-1}}}(W)}{(r_V + r_W - 1)!}(\log q)^{r_V+r_W} q^M M^{r_V+r_W-1}.$$

Noting that $Z_{H_{\omega_{V\times W}^{-1}}}(s) = Z_{H_{\omega_V^{-1}}}(s)Z_{H_{\omega_W^{-1}}}(s)$ together with the fact that V and W satisfy (4.14), we get the expected main term in the asymptotic formula for $N_{V\times W}(M)$ as $M \to \infty$.

It is left to show that the obtained constant agrees with the definition given in Section 4.3. By [65, Lemma 3.0.2], we have $\alpha^*(V)\alpha^*(W) = \alpha^*(V \times W)$ and $\beta(V)\beta(W) = \beta(V \times W)$. This also holds in the case of function fields because α^* and β are geometric invariants. Now let S be a finite subset of the set of places Ω_K of K containing all ramified places and the infinite place. By (5.4), we have that $L_v(1, \operatorname{Pic} V^s)L_v(1, \operatorname{Pic} W^s) = L_v(1, \operatorname{Pic}(V \times W)^s)$ for all $v \in \Omega_K$. We also note that since V and W are projective, we have that $\dim(V \times W) = \dim(V) + \dim(W)$. Thus, $\tau_{H_{\omega_V^{-1}}}(V)\tau_{H_{\omega_W^{-1}}}(W)$ is equal to

$$q^{(1-g_K)(\dim(V\times W))} \prod_{v\in S} \frac{\#\mathcal{V}(\kappa_v)\#\mathcal{W}(\kappa_v)}{(\#k_v)^{\dim(V\times W)}} \prod_{v\notin S} \frac{\#\mathcal{V}(\kappa_v)\#\mathcal{W}(\kappa_v)}{L_v(1,\operatorname{Pic}(V\times W)^s)(\#k_v)^{\dim(V\times W)}},$$

which is exactly $\tau_{H_{\omega_{V\times W}^{-1}}}(V \times W)$. Hence,

$$c_{H_{\omega_V^{-1}}}(V)c_{H_{\omega_W^{-1}}}(W) = \alpha^*(V \times W)\tau_{H_{\omega_{V\times W}^{-1}}}(V \times W) = c_{H_{\omega_{V\times W}^{-1}}}(V \times W),$$

as claimed. □

Lemma 5.3.3. *The number of reducible points in* $\operatorname{Sym}^m \mathbb{P}^2(K)$, *for* $m \geq 2$, *of the shape* $\pi(v_1, \dots, v_m)$, *where* $v_i \in \mathbb{P}^2(K)$ *and* $v_i \neq v_j$ *for all* $1 \leq i, j \leq m$, *is*

$$\sim \frac{S_K(3,1)^m}{3^m m!(m-1)!} q^M M^{m-1},$$

as $M \to \infty$.

Proof. Let $X = V^m$, where V is a Fano variety over a function field K satisfying Manin's conjecture. By Theorem 5.3.2, we have

$$N_X(M) \sim c_{H_{\omega_X^{-1}}}(X) \frac{(\log q)^{mr_V}}{(mr_V - 1)!} q^M M^{mr_V-1},$$

as $M \to \infty$. Set $V = \mathbb{P}^2$. Since $\operatorname{rk}\operatorname{Pic}\mathbb{P}^2 = 1$, we have $\operatorname{rk}\operatorname{Pic} X = m$. Moreover, by Example 2.4.4 and the proof of Theorem 5.3.2 we have

$$c_{H_{\omega_X^{-1}}}(X) = c^m_{H_{\omega_{\mathbb{P}^2}^{-1}}}(\mathbb{P}^2) = \frac{S_K(3,1)^m}{3^m (\log q)^m}.$$

Hence,

$$N_X(M) \sim \frac{S_K(3,1)^m}{3^m(m-1)!} q^M M^{m-1},$$

as $M \to \infty$, which concludes the proof. □

Lemma 5.3.4. *Let $m, t, j \in \mathbb{Z}_{>0}$, such that $t < m$, and $1 \leqslant j < t$. We have*

$$\sum_{i=0}^{M} q^{\frac{(t-j)i}{t}} i^{m-t} = \left(\frac{1}{\frac{t-j}{t}\log q} + \frac{1}{2} \right) q^{\frac{t-j}{t}M} M^{m-t} + O\left(q^{\frac{t-j}{t}M} M^{m-t-1} \right)$$

as $M \to \infty$.

Proof. Let $f(x) = q^{\frac{(t-j)x}{t}} x^{m-t}$. Then, replacing the sum on the left hand-side by an integral we obtain

$$\sum_{i=0}^{M} f(i) \sim \int_0^M f(i)\,\mathrm{d}i + \frac{f(M)}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \left(f^{(2k-1)}(M) - f^{(2k-1)}(0) \right), \tag{5.6}$$

where B_{2k} is the $2k$-th Bernoulli number. We note

$$\int_0^M q^{\frac{(t-j)i}{t}} i^{m-t}\,\mathrm{d}i = \left(\frac{-t}{(t-j)\log q} \right)^{m-t+1} \int_0^{-\log q^{\frac{(t-j)M}{t}}} v^{m-t} e^{-v}\,\mathrm{d}v,$$

via a change of variables. We can write this in terms of gamma functions as

$$\left(\frac{-t}{(t-j)\log q} \right)^{m-t+1} \left(\Gamma(m-t+1) - \Gamma\left(m-t+1, \frac{(t-j)\log q}{-t} M \right) \right).$$

Since $m - t \in \mathbb{Z}_{>0}$, the above is

$$\begin{aligned} &= \left(\frac{-t}{(t-j)\log q} \right)^{m-t+1} (m-t)! \left(1 - q^{\frac{t-j}{t}M} \sum_{k=0}^{m-t} \left(\frac{(t-j)\log q}{-t} \right)^{-k} \frac{M^k}{k!} \right) \\ &= \frac{t}{(t-j)\log q} q^{\frac{t-j}{t}M} M^{m-t} + O\left(q^{\frac{t-j}{t}M} M^{m-t-1} \right). \end{aligned}$$

Computing derivatives of f, we obtain that the sum over k in (5.6) is $O\left(q^{\frac{t-j}{t}M} M^{m-t-1} \right)$. Putting this together leads to the claimed result. □

Lemma 5.3.5. *We have*

$$\frac{1}{k!} \sum_{i=0}^{M} i(M-i)^k \sim \frac{1}{(k+2)!} M^{k+2},$$

as $M \to \infty$.

Proof. By the binomial theorem, the left hand-side is

$$\begin{aligned} \frac{1}{k!} \sum_{i=0}^{M} i \sum_{j=0}^{k} \binom{k}{j} (-i)^j M^{k-j} &= \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^j M^{k-j} \sum_{i=0}^{M} i^{j+1} \\ &\sim \frac{M^{k+2}}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j}{j+2}, \end{aligned}$$

which concludes the proof. □

Proof of Theorem 5.3.1. In the case when $m = 2$, the number of irreducible points of $\mathrm{Sym}^2 \mathbb{P}^2(K)$ is given by (5.3). The reducible points of $\mathrm{Sym}^2 \mathbb{P}^2(K)$ are all of the type in Lemma 5.3.3. Thus, the number of reducible points of $\mathrm{Sym}^2 \mathbb{P}^2(K)$ is

$$\sim \frac{S_K(3,1)^2}{18} q^M M,$$

as $M \to \infty$, which concludes the proof of the case $m = 2$.

Suppose from now that $m \geqslant 3$. The singular points of $\mathrm{Sym}^m \mathbb{P}^2(K)$ are points coming from the diagonal $\Delta \in \left(\mathbb{P}^2\right)^m$, i.e.

$$D(K) = \bigcup_{1 \leqslant i < j \leqslant m} \left\{\pi(v_1, \dots, v_m) \mid v_1, \dots, v_m \in \mathbb{P}^2(K), v_i = v_j\right\},$$

where π is the projection $\left(\mathbb{P}^2\right)^m \to \mathrm{Sym}^m \mathbb{P}^2$ introduced in Section 4.2. If $m \geqslant 3$, by Theorem 5.3.2 , the contribution to $N_{\mathrm{Sym}^m \mathbb{P}^2}(M)$ coming from these points involves counting points $v \in \left(\mathbb{P}^2\right)^{m-2}(K)$ and $v_{m-1} \in \mathbb{P}^2(K)$ such that $H_{\omega^{-1}_{(\mathbb{P}^2)^{m-2}}}(v) \left(H_{\omega^{-1}_{\mathbb{P}^2}}(v_{m-1})\right)^2 = q^M$. Thus we have

$$\begin{aligned} &\sim \left(c_{\omega^{-1}_{\mathbb{P}^2}}(\mathbb{P}^2)\right)^{m-1} q^{\frac{M}{2}} \frac{(\log q)^{m-1}}{2m!(m-3)!} \sum_{i=0}^{M} q^{\frac{i}{2}} i^{m-3}(M-i) \\ &= \frac{S_K(3,1)^{m-1}}{2m!(m-3)!3^{m-1}} q^{\frac{M}{2}} \left(M \sum_{i=0}^{M} q^{\frac{i}{2}} i^{m-3} - \sum_{i=0}^{M} q^{\frac{i}{2}} i^{m-2}\right), \end{aligned}$$

by Example 2.4.4 and Corollary 4.3.2. Now, by Lemma 5.3.4, the terms of order M^{m-2} cancel out and we obtain that the above is

$$\sim \frac{2S_K(3,1)^{m-1}}{3^{m-1} m!(m-3)!(\log q)^2} q^M M^{m-3}.$$

The non-singular points of $\mathrm{Sym}^m \mathbb{P}^2(K)$ are points $\pi(v_1, \dots, v_m)$, where $v_1, \dots, v_m \in \mathbb{P}^2(\overline{K})$ are all distinct and can be partitioned into c_1 Galois conjugacy orbits of length 1, c_2 orbits of length 2, …, c_m orbits of length m, such that $\sum_{i=1}^m ic_i = m$. We remark that a point v_i in a conjugacy orbit of length k is a point in $\mathbb{P}^2(\overline{K})$ such that $[K(v_i) : K] = k$ and the other points in the orbit are its distinct Galois conjugates. It is convenient to analyse all these cases depending on how many orbits of length > 1 there are.

No orbits of length > 1. This implies $c_1 = m$ and , so these are precisely the reducible points of the shape $\pi(v_1, \dots, v_m)$, where $v_i \in \mathbb{P}^2(K)$ and $v_i \neq v_j$ for all $1 \leqslant i, j \leqslant m$, whose contribution is given by Corrolary 5.3.3 and is

$$\sim \frac{S_K(3,1)^m}{3^m m!(m-1)!} q^M M^{m-1},$$

as $M \to \infty$.

One orbit of length > 1. Thus, $c_1 = m - j$ and $c_j = 1$, where $2 \leqslant j \leqslant m$. The case when $j = m$ encompasses exactly the irreducible points and their contribution is given by (5.3). Now fix j such that $2 \leqslant j < m$. These are points $\pi(v_1, \dots, v_m)$ such that $v_1, \dots, v_{m-j} \in \mathbb{P}^2(K)$, $v_{m_j+1} \in \mathbb{P}^2(\overline{K})$, $[K(v_{m_j+1}) : K] = j$, and $v_{m_j+1}, \dots, v_m$ are the j distinct Galois conjugates of v_{m_j+1}. Counting such points with height $H_{\omega^{-1}_{\mathrm{Sym}^m \mathbb{P}^2}}(\pi(v_1, \dots, v_m)) = q^M$ corresponds to counting $v \in \left(\mathbb{P}^2\right)^{m-j}(K)$ with $H_{\omega^{-1}_{(\mathbb{P}^2)^{m-j}}}(v) = q^i$ and $w \in \mathbb{P}^2(\overline{K})$ such that $[K(w) : K] = j$ and $H_{\omega^{-1}_{\mathbb{P}^2}}(w)^j = q^{M-i}$, for $0 \leqslant i \leqslant M$. Thus, it is equal to

$$\frac{j!}{m!} \sum_{i=0}^{M} N_{(\mathbb{P}^2)^{m-j}}(i) N_{\mathrm{Sym}^j \mathbb{P}^{2\,\mathrm{irr}}} \left(\frac{M-i}{j}\right). \tag{5.7}$$

By Manin's conjecture for the irreducible points in $\mathrm{Hilb}^m \mathbb{P}^2$ and Theorem 5.3.2, we expect that, as $L \to \infty$, there are

$$\sim \left(c_{\omega^{-1}_{\mathbb{P}^2}}\left(\mathbb{P}^2\right)\right)^{m-j} c_{\omega^{-1}_{\mathrm{Hilb}^j \mathbb{P}^2}}\left(\mathrm{Hilb}^j \mathbb{P}^2\right) \frac{(\log q)^{m-j+2}}{(m-j+1)!} q^L L^{m-j+1}$$

points $(v, w) \in \left(\mathbb{P}^2\right)^{m-j} \times \mathrm{Sym}^j \mathbb{P}^2(K)^{\mathrm{irr}}$ of height $H_{\omega^{-1}_{(\mathbb{P}^2)^{m-j}}}(v) H_{\omega^{-1}_{\mathrm{Sym}^j \mathbb{P}^2}}(w) = q^L$, since the rank of the Picard group of the product variety is $m - j + 2$. If $2 < j < m$, this is at most $\sim L^{m-2}$, and only the case $j = 2$ gives $\sim L^{m-1}$. This implies that the number of points given by (5.7) is at most $O\left(M^{m-2}\right)$ for $2 < j < m$, and thus, does not contribute to the main term in $N_{\mathrm{Sym}^m \mathbb{P}^2}(M)$. In the case when $j = 2$, we obtain that that (5.7) is

$$\begin{aligned}
&\sim \frac{2}{m!}\left(c_{\omega^{-1}_{\mathbb{P}^2}}(\mathbb{P}^2)\right)^{m-2} c_{\omega^{-1}_{\mathrm{Hilb}^2 \mathbb{P}^2}}\left(\mathrm{Hilb}^2 \mathbb{P}^2\right) \frac{(\log q)^m}{(m-3)!} \sum_{i=0}^{M} q^{i+\frac{M-i}{2}} i^{m-3} \frac{M-i}{2} \\
&= \frac{S_K(3,1)^m}{3^m m!(m-3)!} q^{\frac{M}{2}} \left(M \sum_{i=0}^{M} q^{\frac{i}{2}} i^{m-3} - \sum_{i=0}^{M} q^{\frac{i}{2}} i^{m-2}\right),
\end{aligned}$$

by Example 2.4.4 and Corollary 4.3.2. Now, by Lemma 5.3.4, the terms of order M^{m-2} cancel out and we obtain that the above is

$$\sim \frac{4 S_K(3,1)^m}{3^m m!(m-3)!\,(\log q)^2} q^M M^{m-3}.$$

Through a similar method we can show that the contribution from the cases when $2 < j < m$ is in fact at most $O\left(M^{m-4}\right)$.

k orbits of length > 1*, where* $1 < k \leqslant \lfloor m/2 \rfloor$. This is a generalisation of the previous case. Thus, we have $c_1 = m - 2k$ and $\sum_{i=2}^{m} c_i = k$. We expect that the number of points

$x \in \left(\mathbb{P}^2\right)^{m-2k}(K) \times \left(\operatorname{Sym}^2 \mathbb{P}^2\right)^{j_1}(K)^{\text{irr}} \ldots \times \left(\operatorname{Sym}^m \mathbb{P}^2\right)^{j_m}(K)^{\text{irr}}$, where $\sum_{i=2}^m i j_i = 2k$, of height $H(x) = q^L$, where H is the product of the anticanonical heights, is

$$\left(c_{\omega^{-1}_{\mathbb{P}^2}}(\mathbb{P}^2)\right)^{m-2k} \prod_{i=2}^m \left(c_{\omega^{-1}_{\operatorname{Hilb}^i \mathbb{P}^2}}\left(\operatorname{Hilb}^i \mathbb{P}^2\right)\right)^{j_i} \frac{(\log q)^{m-2k+2l}}{(m-2k+2l-1)!} q^L L^{m-2k+2l-1},$$

as $L \to \infty$, since the rank of the Picard group of the product variety is $m-2k+2l$, where $l = j_2 + \ldots + j_m$. A simple calculation shows that only in the case when $j_2 = k$ and $j_3 = \ldots = j_m = 0$ the above is $\sim L^{m-1}$ and in all other cases we have at most L^{m-2} points. Thus we analyse the former case. The number of such points with height $H_{\omega^{-1}_{\operatorname{Sym}^m \mathbb{P}^2}}(\pi(v_1, \ldots, v_m)) = q^M$ is

$$\begin{aligned} &\frac{2^k}{m!} \sum_{i=0}^{M} N_{(\mathbb{P}^2)^{m-2k}}(i) \sum_{i_1=0}^{M-i} N_{\operatorname{Sym}^2 \mathbb{P}^{2\text{irr}}}\left(\frac{i_1}{2}\right) \ldots \sum_{i_{k-1}=0}^{M-i-i_1-\ldots i_{k-2}} \\ &\times N_{\operatorname{Sym}^2 \mathbb{P}^{2\text{irr}}}\left(\frac{i_{k-1}}{2}\right) N_{\operatorname{Sym}^2 \mathbb{P}^{2\text{irr}}}\left(\frac{M-i-i_1-\ldots-i_{k-1}}{2}\right). \end{aligned} \tag{5.8}$$

Denote $M - i - i_1 - \ldots - i_{k-2}$ by L_1. Then, the last sum in (5.8) is

$$\begin{aligned} &\sim \left(c_{\omega^{-1}_{\operatorname{Hilb}^2 \mathbb{P}^2}}\left(\operatorname{Hilb}^2 \mathbb{P}^2\right)\right)^2 \frac{(\log q)^4}{2^2} q^{\frac{L_1}{2}} \sum_{i=0}^{L_1} i_{k-1}(L_1 - i_{k-1}) \\ &\sim \left(c_{\omega^{-1}_{\operatorname{Hilb}^2 \mathbb{P}^2}}\left(\operatorname{Hilb}^2 \mathbb{P}^2\right)\right)^2 \frac{(\log q)^4}{2^2} q^{\frac{L_1}{2}} \frac{{L_1}^3}{3!}, \end{aligned}$$

by Lemma 5.3.5. We iterate this procedure for the sums in (5.8) starting with the sum over i_{k-2} up to the sum over i_{i_1} to obtain that (5.8) is

$$\begin{aligned} &\sim \frac{2^k}{m!} \sum_{i=0}^{M} N_{(\mathbb{P}^2)^{m-2k}}(i) \left(c_{\omega^{-1}_{\operatorname{Hilb}^2 \mathbb{P}^2}}\left(\operatorname{Hilb}^2 \mathbb{P}^2\right)\right)^k \frac{(\log q)^{2k}}{2^k} q^{\frac{M-i}{2}} \frac{(M-i)^{2k-1}}{(2k-1)!} \\ &\sim \frac{\left(c_{\omega^{-1}_{\mathbb{P}^2}}(\mathbb{P}^2)\right)^{m-2k} \left(c_{\omega^{-1}_{\operatorname{Hilb}^2 \mathbb{P}^2}}\left(\operatorname{Hilb}^2 \mathbb{P}^2\right)\right)^k (\log q)^m q^{\frac{M}{2}}}{m!(m-2k-1)!(2k-1)!} \sum_{i=0}^{M} q^{\frac{i}{2}} i^{m-2k-1} (M-i)^{2k-1}. \end{aligned}$$

By the binomial theorem, the sum over i above is equal to

$$\begin{aligned} &= \sum_{j=0}^{2k-1} \binom{2k-1}{j} (-1)^j M^{2k-1-j} \sum_{i=0}^{M} q^{\frac{i}{2}} i^{m-2k-1+j} \\ &= \left(\frac{2}{\log q} + \frac{1}{2}\right) q^{\frac{M}{2}} M^{m-2} \sum_{j=0}^{2k-1} \binom{2k-1}{j} (-1)^j + O\left(q^{\frac{M}{2}} M^{m-3}\right), \end{aligned}$$

by Lemma 5.3.4. Noting that $\sum_{j=0}^{2k-1} \binom{2k-1}{j}(-1)^j = 0$, we get a contribution $O\left(M^{m-3}\right)$. Through a similar method, we can show that the contribution coming from the other choices of c_i's is in fact at most $O\left(M^{m-4}\right)$.

In conclusion, the main contribution to $N_{\operatorname{Sym}^m \mathbb{P}^2}(M)$ comes from reducible points of the type described in Lemma 5.3.3. $\square$

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